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ABSTRACT

This is part two of a two-part SMSG text for grade eight students whose mathematical talents are underdeveloped. The reading level of this text has been adjusted downward, chapters shortened, and additional concrete examples included. Nevertheless, the authors warn that the text may not be appropriate for the very slow non-college-bound student. Chapter topics include: negative rational numbers, equations and inequalities, coordinates in the plane, real numbers, and scientific notation, decimals, and the metric system. (MN)

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# Introduction to Secondary School Mathematics, Volume 2

## *Student's Text, Part II*

REVISED EDITION

Prepared under the supervision of
a Panel consisting of:

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# CONTENTS

Chapter

17. NEGATIVE RATIONAL NUMBERS . . . . . . . . . . . . . . 239

- 17-1. Introduction . . . . . . . . . . . . . . . . . 239
- 17-2. The Negative Half-Line . . . . . . . . . . 242
- 17-3. Addition on the Number Line . . . . . . . 247
- 17-4. Subtraction of Rational Numbers . . . . . 254
- 17-5. Multiplication of Rational Numbers . . . 260
- 17-6. Division of Rational Numbers . . . . . . . 268
- 17-7. Summary . . . . . . . . . . . . . . . . . . 272
- 17-8. Chapter Review . . . . . . . . . . . . . . . 273
- 17-9. Cumulative Review . . . . . . . . . . . . . 275

18. EQUATIONS AND INEQUALITIES . . . . . . . . . . . . 279

- 18-1. Sentences and Phrases . . . . . . . . . . 279
- 18-2. Sentences and Their Solutions . . . . . . 283
- 18-3. Formulas . . . . . . . . . . . . . . . . . . 287
- 18-4. Graphing Solution Sets of Sentences . . . 292
- 18-5. Summary . . . . . . . . . . . . . . . . . . 296
- 18-6. Chapter Review . . . . . . . . . . . . . . . 297
- 18-7. Cumulative Review . . . . . . . . . . . . . 298

19. COORDINATES IN THE PLANE . . . . . . . . . . . . . 301

- 19-1. Locating Points in a Plane . . . . . . . . 301
- 19-2. Coordinates in the Plane . . . . . . . . . 306
- 19-3. Graphs in the Plane . . . . . . . . . . . 316
- 19-4. Other Equations, Other Lines . . . . . . . 321
- 19-5. Distance . . . . . . . . . . . . . . . . . . 330
- 19-6. A Property of Right Triangles . . . . . . 335
- 19-7. A Proof of the Pythagorean Property . . . 338
- 19-8. Back to Distance . . . . . . . . . . . . . 343
- 19-9. Summary . . . . . . . . . . . . . . . . . . 350
- 19-10. Chapter Review . . . . . . . . . . . . . . 351
- 19-11. Cumulative Review . . . . . . . . . . . . 352
- Table--Squares and Square Roots of Numbers . . . . . . . . . . . . . . . . 354

20. REAL NUMBERS . . . . . . . . . . . . . . . . . . . 357

- 20-1. Real Numbers as Points on the Number Line . . . . . . . . . . . . . . . 357
- 20-2. Locating Numbers on the Number Line . . . 360
- 20-3. Irrational Numbers . . . . . . . . . . . . 362
- *20-4. Irrational Numbers and Infinite Decimals . . . . . . . . . . . . . . . . . 366
- 20-5. Properties of Number Systems . . . . . . . 373
- 20-6. Summary . . . . . . . . . . . . . . . . . . 376
- 20-7. Chapter Review . . . . . . . . . . . . . . . 376
- 20-8. Cumulative Review . . . . . . . . . . . . . 377

21. SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM . . . . . . . . . . . . . . . . . . . 379

21-1. Large Numbers and Scientific Notation . 379
21-2. Multiplying by Powers of Ten . . . . . . . 383
21-3. Dividing by Powers of Ten . . . . . . . . 386
21-4. Multiplication of Large and Small Numbers . . . . . . . . . . . . . . . . . . 393
21-5. Division of Large and Small Numbers . . 396
21-6. Use of Exponents in Multiplying and Dividing Decimals . . . . . . . . . . . . 399
21-7. The Metric System . . . . . . . . . . . . 403
21-8. Conversion to English Units . . . . . . . 407
21-9. Summary . . . . . . . . . . . . . . . . . . 410
21-10. Chapter Review . . . . . . . . . . . . . . 410
21-11. Cumulative Review . . . . . . . . . . . . 412
Index . . . . . . . . . . following page 413

## Chapter 17

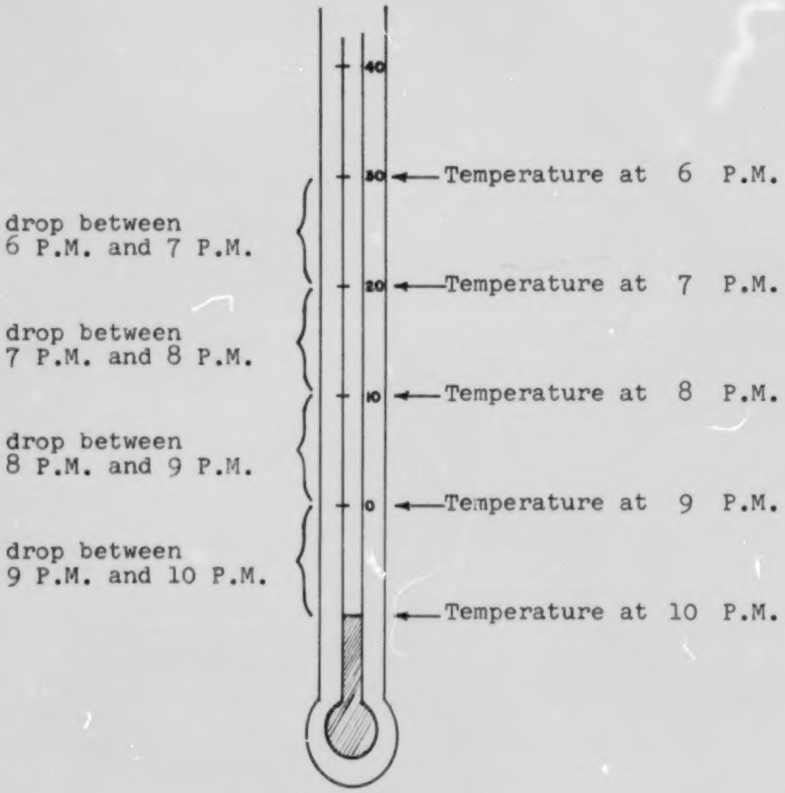
## NEGATIVE RATIONAL NUMBERS

### 17-1. Introduction.

You will often see situations in which on one line there are two number scales extending in opposite directions from the origin. Here is an example.

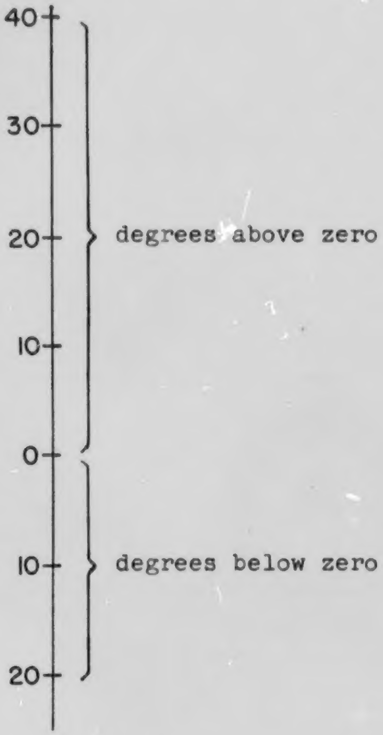
Example 1: One January night in Minneapolis the temperature was 30 degrees at 6 P.M. Then, a terrible storm came up which was not predicted by the weather bureau and the temperature dropped 10 degrees each hour for 4 hours. What was the temperature at 10 P.M?

Let us see how this looks on the thermometer.


40
30
20
10
0
Temperature at 6 P.M.
Temperature at 7 P.M.
Temperature at 8 P.M.
Temperature at 9 P.M.
Temperature at 10 P.M.
drop between 6 P.M. and 7 P.M.
drop between 7 P.M. and 8 P.M.
drop between 8 P.M. and 9 P.M.
drop between 9 P.M. and 10 P.M.

The last temperature drop carried us over onto the other side of zero on the thermometer. Though it has not been labeled yet in our picture, you would say that the temperature was 10 degrees below zero at 10 P.M.

A thermometer can be thought of as a number line drawn vertically, with numbers on both sides of zero.

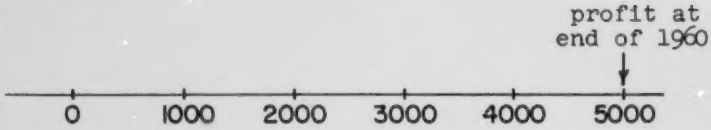


A similar thing happens in this example.

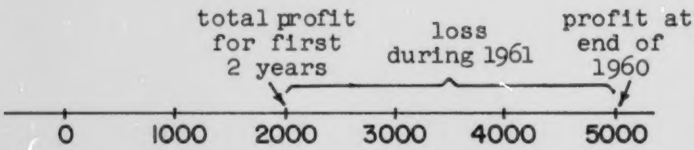
Example 2: A certain department store started a transistor radio department in 1960. During that year, it made $5000 profit. During 1961, the department lost $3000 and during 1962, it lost $4000. What was the total profit for the three years?

If we illustrate this problem on the number line, we have a picture like this.

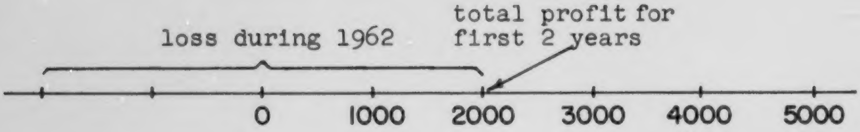
First Step.


profit at end of 1960
0
1000
2000
3000
4000
5000

Second Step.

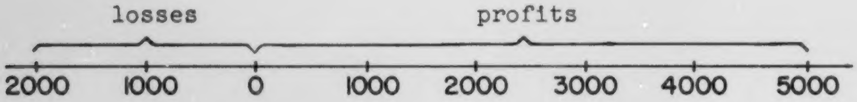

total profit for first 2 years
loss during 1961
profit at end of 1960
0
1000
2000
3000
4000
5000

Third Step.


loss during 1962
total profit for first 2 years
0
1000
2000
3000
4000
5000

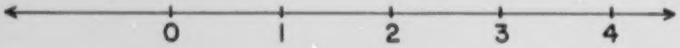
We see that the financial position of the department at the end of the third year will be represented by a point on the other side of zero. We can see that this point corresponds to a loss (or deficit) of $2000.

In business problems as well as in temperature, there is need for number lines with numbers indicated on both sides of zero.


losses
profits
2000
1000
0
1000
2000
3000
4000
5000

These are just two of many examples which call for number lines with points on both sides of the origin corresponding to numbers.

In our earlier work, we have often referred to the number line, but we have really used only a number ray. This ray consists of the origin (the point labeled "0") and all the points on one side of the origin.



Now, we are ready to consider points on the other side of the origin. We shall extend our number system to include numbers represented by points on this other half-line. When we have done this, we shall have a true number line.

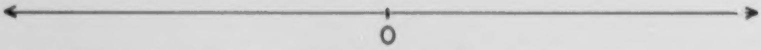
---

## 17-2. The Negative Half-Line.

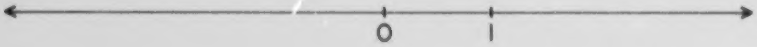
In the preceding section, you were told that points on the number line on both sides of the origin will be used to represent numbers. In order to do this, we invent some new numbers. In this section, we shall see how these new numbers are named. First, we recall how the number line is formed.

In forming the number line, we first select a line.

On this line, we choose an arbitrary point to represent the number 0. Remember that we call this point the origin.

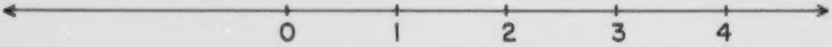


Next, we choose another point arbitrarily to represent the number 1.

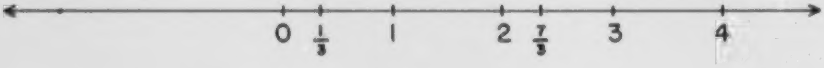


After this, we have no freedom of choice. The location of all other numbers on the number line is now strictly determined.

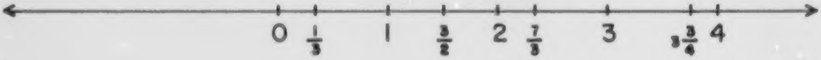
All the numbers represented by points on the same side of the origin as the number 1 are called positive numbers. Thus, all the numbers we have met in earlier chapters (except 0 itself) are positive. You should recall how these numbers can be located. The other whole numbers are located by marking off the distance between 0 and 1 over and over again.



The number $\frac{1}{3}$ is located one-third of the way from 0 to 1. The number $\frac{7}{3}$ can be expressed as $2\frac{1}{3}$ and then located as shown.



Similarly, fractions with other denominators are located. Some are shown below.



Remember that we say that $\frac{3}{2} < \frac{7}{3}$ (read, $\frac{3}{2}$ is less than $\frac{7}{3}$) because $\frac{3}{2}$ lies to the left of $\frac{7}{3}$ on the number line. We could also say that $\frac{7}{3} > \frac{3}{2}$ ($\frac{7}{3}$ is greater than $\frac{3}{2}$).

## Exercises 17-2a

1. Draw a number line and locate the following numbers on it.

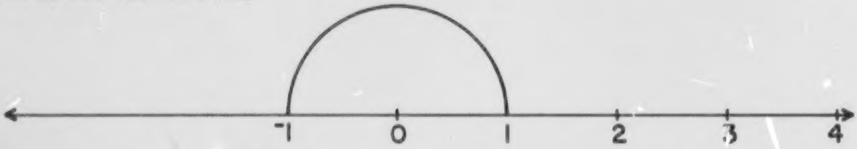
   $\frac{4}{2}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{6}{5}$, $\frac{12}{4}$, $\frac{7}{11}$, $\frac{3}{7}$, $\frac{8}{2}$

2. Replace the question mark by $>$, $<$ or $=$ in each of the following by using the picture in Problem 1.

   a. $\frac{4}{2} ? \frac{4}{3}$  b. $\frac{3}{2} ? \frac{6}{5}$  c. $\frac{8}{2} ? \frac{3}{7}$  d. $\frac{7}{4} ? \frac{12}{4}$.

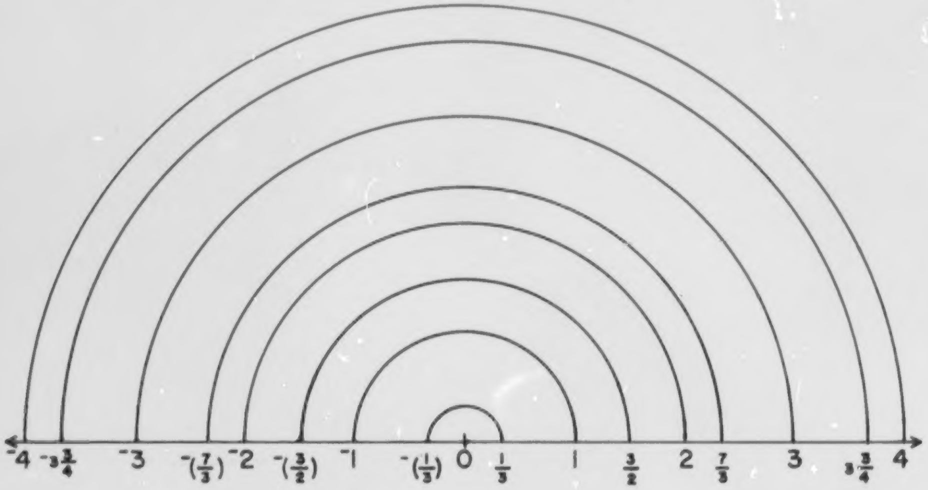
3. Check each of your answers in Problem 2 by using the Comparison Property.

Now, we are ready to name some points on the other side of the origin. Suppose we place a compass with the needle at the origin and the pencil point at 1, and then draw a half-circle as shown.



In this way, we locate a point on the other side of zero on the number line. This point represents a number which we call negative one, written "$^-1$". (A raised bar is placed in front of the symbol "1".) All numbers located on this side are called negative numbers.

The way in which some other points on the number line are located and named is shown in the figure below.



We see that the numbers 1 and $^{-}1$ are located opposite to each other on the number line. That is, they are on opposite sides of the origin and both at the same distance from the origin. We say that $^{-}1$ is the *opposite* of 1 and that 1 is the opposite of $^{-}1$. Similarly, $^{-}2$ is the opposite of 2 and $\frac{3}{2}$ is the opposite of $^{-}(\frac{3}{2})$.

We see that the number 0 separates the number lines into two half-lines called the *positive half-line* and the *negative half-line*. Numbers on the negative half-line are called *negative numbers*. Numbers on the positive half-line are called *positive numbers*. The number 0 is neither positive nor negative. Notice that the opposite of a positive number is a negative number and the opposite of a negative number is a positive number. The opposite of zero is zero.

All the numbers we have studied in earlier chapters were called rational numbers. All these numbers, except for the number zero, are positive and we will therefore call them *positive rational numbers*. The negatives of the positive rational numbers will be called *negative rational numbers*. From now on, when we speak of the *set of rational numbers*, we mean the set consisting of the positive rational numbers, the negative rational numbers and the number zero.

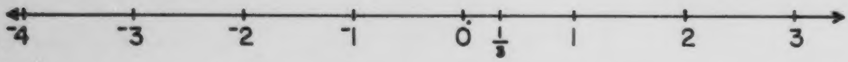
An especially important subset of the set of rational numbers is the set of *integers*. This set consists of the number zero, the counting numbers, and the negatives of the counting numbers. The set of integers may be expressed as:

$$\{\ldots, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, \ldots\}.$$

The set of counting numbers may also be called the set of positive integers. The set of whole numbers may be referred to as the set of non-negative integers.

When we say that a is less than b, we mean that a is to the left of b on the number line.

From the figure below, we see that $^{-}4 < {}^{-}3$ and that $^{-}1 < \frac{1}{3}$.



In fact, every negative number is less than any positive number.

## Exercises 17-2b

(Class Discussion)

1. For each of the following, tell whether you would represent it by a positive or a negative number.

   a. 20 degrees below zero.

   b. 20 degrees above zero.

   c. $300 profit in business.

   d. $300 deficit in business.

   e. 50 feet below sea level.

   f. 50 feet above sea level.

   g. a 4-yard loss in football.

   h. a 4-yard gain in football.

2. Give another name for each of the following:

   a. the set of rational numbers greater than zero.
   b. the set of rational numbers less than zero.
   c. the set of integers greater than zero.
   d. the set of integers less than zero.

3. Find the opposite of each number given below.

   a. $4$

   b. $^{-}8$

   c. $\frac{2}{3}$

   d. $^{-}2\frac{1}{2}$

   e. $^{-}(\frac{5}{4})$

   f. $5.3$

   g. $^{-}6.4$

   h. $0$

4. List the following numbers in the order in which they appear on the number line.

   $-4, \frac{1}{4}, {}^{-}(\frac{7}{4}), \frac{5}{8}, {}^{-}6, {}^{-}(\frac{3}{8}), \frac{3}{4}.$

Which is the greatest of these numbers? Which is the least?

## Exercises 17-2c

1. Draw a number line and locate the following numbers.

$$^{-}3\frac{1}{2},\ \frac{2}{5},\ \frac{3}{2},\ ^{-}\left(\frac{14}{3}\right),\ 2.3,\ ^{-}1.8$$

2. Using the picture in Problem 1, copy the following and replace "?" by the correct symbol ">" or "<".

   a. $^{-}3\frac{1}{2}\ ?\ \frac{3}{2}$

   b. $\frac{2}{5}\ ?\ \frac{3}{2}$

   c. $^{-}3\frac{1}{2}\ ?\ ^{-}1.8$

   d. $^{-}3\frac{1}{2}\ ?\ ^{-}\left(\frac{14}{3}\right)$

   e. $\frac{2}{5}\ ?\ ^{-}1.8$

   f. $^{-}1.8\ ?\ 2.3$

3. (Matching) For each number in the first list, write down the letter corresponding to its opposite in the second list.

   a. $\frac{2}{3}$

   b. $\frac{3}{5}$

   c. $^{-}2\frac{1}{2}$

   d. $1.6$

   e. $^{-}\left(\frac{7}{3}\right)$

   f. $^{-}\left(\frac{9}{4}\right)$

   A. $^{-}\left(\frac{8}{5}\right)$

   B. $\frac{5}{2}$

   C. $2\frac{1}{3}$

   D. $^{-}\left(\frac{2}{3}\right)$

   E. $2.25$

   F. $^{-}(.6)$

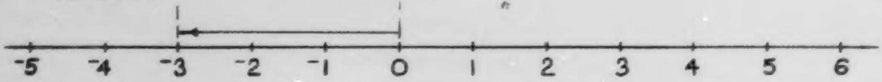
---

## 17-3. Addition on the Number Line.

You know how to draw arrows on the number line to represent positive numbers. These arrows are drawn starting at zero and ending at the point which corresponds to the number. (We draw the arrow directly above the number line instead of on it so it can be seen clearly.)

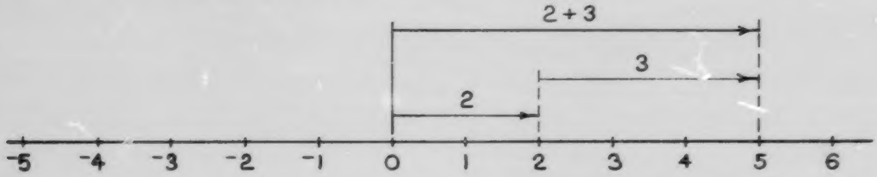
We use a similar procedure to represent a negative number.

Represent $^{-}3$ on the number line. The point $^{-}3$ is three units to the left of 0. An arrow beginning at 0 and marked off three units to the left represents this number.

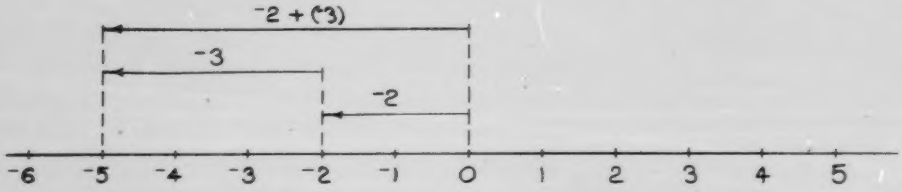


You have drawn diagrams showing addition of positive numbers. Now, we can do the same for all rational numbers.

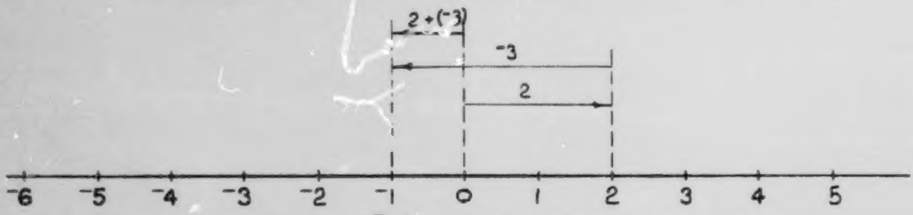
**Example 1**: First, review how to represent 2 + 3 on the number line. We represent "2" by an arrow starting at 0 and ending at the point 2. The arrow representing "3" starts at the head of the "2" arrow and is marked off 3 units to the right. The sum "2 + 3" is represented by an arrow starting at 0 and ending at the head of arrow "3".



**Example 2**: Represent $^{-}2 + (^{-}3)$ on the number line. We diagram this problem using the same method as in Example 1, remembering that the arrows representing negative numbers always point to the left. $^{-}2 + (^{-}3) = {}^{-}5$.

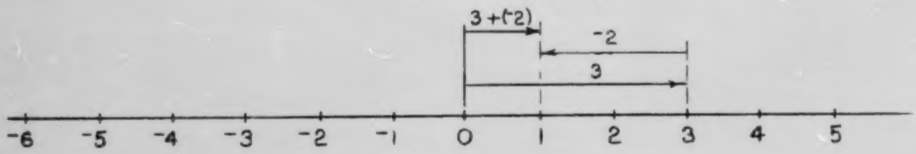


Example 3: Represent $2 + (^-3)$ on the number line. In this example the arrow representing "2" will begin at 0 and end at the point 2. The arrow representing "$^-3$" will begin at the head of the arrow "2", but will go 3 units to the left.



The sum $2 + (^-3)$ is indicated by an arrow beginning at 0 and ending at the head of the arrow representing "$^-3$". $2 + (^-3) = {}^-1$.

Example 4: Represent $3 + (^-2)$ on the number line. In this example, the arrow for "$^-2$" is marked off to the left starting at the head of arrow "3". Again, the sum is indicated by an arrow beginning at 0 and ending at the head of the arrow representing "$^-2$". $3 + (^-2) = 1$.



## Exercises 17-3a
(Class Discussion)

1. In which direction does an arrow point if it represents a positive number?

2. In which direction does an arrow point if it represents a negative number?

3. Describe the arrangement of the arrows when all numbers to be added are positive.

4. Describe the arrangement of arrows when all addends are negative.

5. Describe the arrangement when one number is positive and one is negative. In what direction will the arrow representing the sum point?

6. How can you show three or more addends on the number line?

7. How can you show $3 + 0$?

## Exercises 17-3b

1. Draw a diagram on a number line to find the sum in each of the following problems.

   a. $2 + 4$
   b. $1 + 6$
   c. $^{-}5 + (^{-}2)$
   d. $^{-}4 + (^{-}4)$
   e. $4 + (^{-}4)$
   f. $^{-}3 + 5$
   g. $4 + (^{-}1)$
   h. $2 + (^{-}6)$
   i. $^{-}7 + 3$
   j. $^{-}6 + 6$

2. Find the missing number. You may use a number line to help you but you do not need to show it on your paper.

   a. $3 + (^{-}3) = (\quad)$
   b. $(\quad) + (^{-}4) = 0$
   c. $6 + (\quad) = 0$
   d. $^{-}75 + 74 = (\quad)$
   e. $14 + (^{-}2) = (\quad)$
   f. $\frac{1}{3} + (\quad) = 1$
   g. $^{-}(\frac{1}{2}) + (\quad) = 0$
   h. $^{-}0.45 + 0.45 = (\quad)$

3. What number must be added to each of these numbers to have a sum of zero?

   a. $5$
   b. $^{-}3$
   c. $^{-}(\frac{3}{2})$
   d. $\frac{3}{4}$
   e. $^{-}2$
   f. $4$
   g. $^{-}25$
   h. any positive number, $a$

4. Some ideas are represented by positive numbers and others by negative. Use "⁻" for ideas represented by negative numbers and "+" for ideas represented by positive numbers.

   a. Profit

   b. Loss

   c. If direction east is positive, west is ______?

   d. North

   e. South

   f. Gaining yards in football

   g. A loss in football

   h. Altitude above sea level

   i. Altitude below sea level

   j. Time in the future

   k. Time in the past

5. A company report of profit or loss was the following for the first 6 months of a year:

| | | | |
|---|---|---|---|
| January | $5000 profit | April | $1000 profit |
| February | $2400 profit | May | $3000 loss |
| March | $1000 loss | June | $2000 loss |

   a. Represent each of these figures as a positive or negative number.

   b. Find the total profit or loss for all six months.

   c. What is the average profit or loss per month?

6. BRAINBUSTER: An escalator going down moves at the rate of 1 step per second. A boy can run up steps at the rate of 2 steps per second. A boy runs up an escalator that is going down. If there are 28 steps between floors, how long will it take him to go up one floor? If a second boy can run up 3 steps in one second, how long will he take?

## Exercises 17-3c

In Problems 1 to 3, add without using the number line.

1. a. $4\frac{3}{5} + 5\frac{4}{5}$

   b. $5\frac{4}{5} + 4\frac{3}{5}$

   c. $(2 + 3) + 5$

   d. $2 + (3 + 5)$

   e. $7\frac{1}{2} + 0$

2. a. $^-1\frac{1}{2} + (^-2\frac{1}{2})$

   b. $^-1\frac{1}{3} + (^-3\frac{1}{3})$

   c. $^-1\frac{5}{6} + (^-4\frac{1}{2})$

   d. $^-4\frac{1}{2} + (^-1\frac{5}{6})$

   e. $[^-2 + (^-1\frac{1}{2})] + (^-3\frac{3}{4})$

   f. $^-2 + [^-1\frac{1}{2} + (^-3\frac{3}{4})]$

   g. $0 + (^-4\frac{1}{3})$

3. a. $3\frac{1}{2} + (^-4\frac{1}{2})$

   b. $^-4\frac{1}{3} + 2\frac{1}{3}$

   c. $4\frac{3}{5} + (^-5\frac{4}{5})$

   d. $^-5\frac{4}{5} + 4\frac{3}{5}$

   e. $^-4\frac{3}{5} + 5\frac{4}{5}$

   f. $[2\frac{1}{2} + (^-\frac{3}{4})] + 4$

   g. $2\frac{1}{2} + [(^-\frac{3}{4}) + 4]$

   h. $[(^-2\frac{1}{2}) + \frac{3}{4}] + (^-4)$

   i. $(^-5\frac{1}{2}) + 5\frac{1}{2}$

4. In your own words, describe how you would add rational numbers if:

   a. both are positive.

   b. both are negative.

   c. one is positive and one is negative.

The number line has been used to help you see how to add positive and negative numbers. You should observe that the properties that are true for the addition of non-negative rational numbers are still true for the entire set of rational numbers.

We see by the use of the number line that the addition of numbers, whether positive or negative, is really very simple. We need only keep in mind the location of the numbers on the number line to carry out the operation.

We see that:

When both numbers are positive the sum is positive,
as in $2 + 3 = 5$;

and, when both numbers are negative the sum is negative,
as in $^{-}2 + (^{-}3) = {}^{-}5$.

When one number is positive and one number is negative, it is the number farther from the origin which determines whether the sum is positive or negative.

For example:

In $2 + (^{-}3) = {}^{-}1$,

the sum is negative because $^{-}3$, which is farther from zero than 2, is negative.

In $3 + (^{-}2) = 1$,

the sum is positive, because 3, which is farther from zero than $^{-}2$, is positive.

We can say this in another way. The arrow of greater length determines whether the sum is positive or negative.

Notice that in cases like $^{-}2 + 2$ and $3 + (^{-}3)$, the arrows are of equal length but opposite in direction. In these cases, the sum is zero.

### Exercises 17-3d

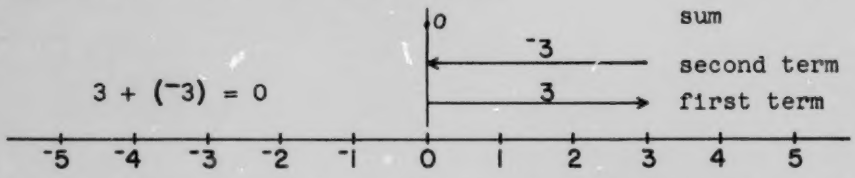
(Class Discussion)

1. Find examples of each of these properties in Exercises 17-3c.
   a. The commutative property for addition.
   $$a + b = b + a$$
   b. The associative property for addition.
   $$a + (b + c) = (a + b) + c$$
   c. The addition property of zero.
   $$a + 0 = a$$
2. Add:
   a. $2 + (^{-}2)$
   b. $3\frac{1}{2} + (^{-}3\frac{1}{2})$
   c. $^{-}(\frac{4}{3}) + \frac{4}{3}$

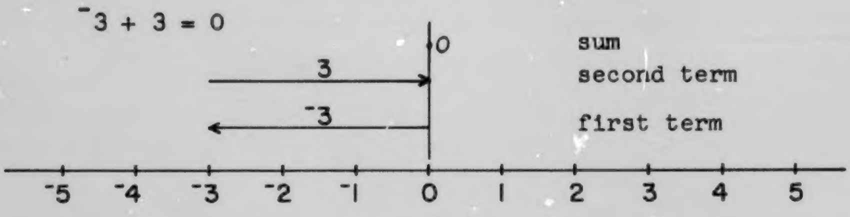
3. What do you notice about the numbers being added and the answers in each part of Problem 2?

4. Do the familiar properties of addition hold for all rational numbers?

We have seen in the above exercises that the sum of a number and its opposite is equal to zero. For example: $3 + (^-3) = 0$ or $^-3 + 3 = 0$. These addition problems are illustrated in the following diagrams.



The sum is indicated this time by a point (an arrow drawn from zero to zero has no length). Similarly



Two numbers whose sum is zero are said to be additive inverses of each other. We see that this means the same thing as saying that they are opposites of each other. Thus, $^-3$ is the additive inverse of 3 and 3 is the additive inverse of $^-3$. These statements merely say that

$$3 + (^-3) = {^-3} + 3 = 0.$$

## 17-4. Subtraction of Rational Numbers.

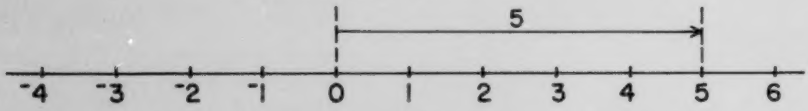
Consider the subtraction problem, $5 - 3$. You are looking for the number $n$ which, when added to 3, gives 5.

$$3 + n = 5$$

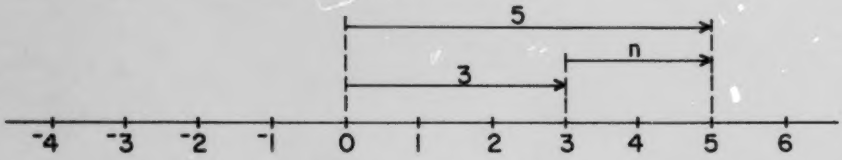
Let us review how we diagram this subtraction problem on the number line.

**Example 1:** $5 - 3 = n$ or $3 + n = 5$

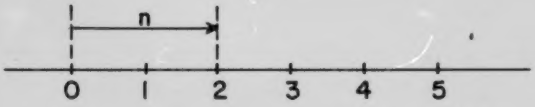
First, we represent "5" by an arrow which starts at 0 and ends at point 5.



Next, we represent "3" by the arrow which starts at 0 and ends at point 3.



Now what number must be added to 3 to give 5? This will be the number n represented by the arrow which starts at the end of arrow 3 and ends at the end of arrow 5. To find the number n, we must transfer arrow n so that it begins at 0, then the arrowhead will fall on the point associated with n.



In this case n is 2. (You may be able to do this step of transferring the arrow mentally.)

Let us look at another subtraction problem, 3 - 5. Here, 3 - 5 must be the number n which added to 5, gives 3.

$$n = 3 - 5 \quad \text{or} \quad 5 + n = 3.$$

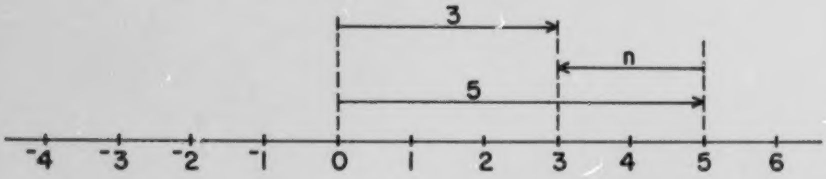
You cannot find such a number in the set of positive rational numbers. Remember your teachers in the earlier grades were careful not to ask subtraction problems of this type, where a greater number is subtracted from a smaller number. Now that you have extended the set of numbers to include the negative rational numbers, let us see if such subtraction problems can be solved. This can be shown on a diagram.

**Example 2:** $3 - 5 = n$ or $5 + n = 3$

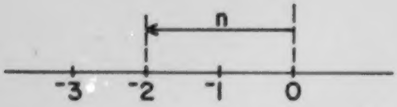
a. First represent the sum $3$ as an arrow starting at $0$.

b. Then, represent $5$ as an arrow starting at $0$.

c. Then, draw an arrow starting at the head of arrow $5$ and ending at the head of arrow $3$. This is arrow $n$.

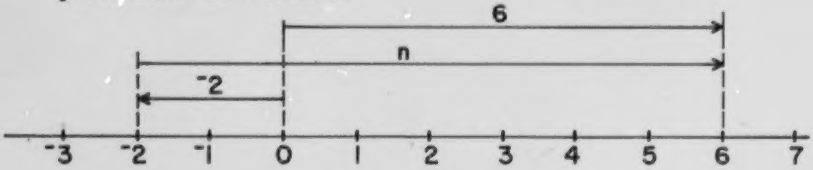


d. To find the number that arrow $n$ represents, we transfer the arrow so that it begins at zero. We read the number at the head of this arrow. $n$ is $^{-}2$.



We see $n = 3 - 5 = {}^{-}2$.

**Example 3:** What number is $6 - ({}^{-}2)$? It is the number $n$ so that ${}^{-}2 + n = 6$. We shall find n by the same procedure as before.



From this diagram, you can see $n = 6 - ({}^{-}2) = 8$.

From these examples, you should notice several things.

1. Each subtraction problem can be restated as an addition problem. As soon as we introduce the negative rational numbers, every subtraction problem involving

rational numbers has a solution. The set of rational numbers is closed under subtraction.

2. When we diagram subtraction problems on the number line in this way, the direction of the arrow which represents the difference tells us whether the difference is positive or negative.

3. The length of the arrow is equal to the distance between the origin and this difference.

## Exercises 17-4a
(Class Discussion)

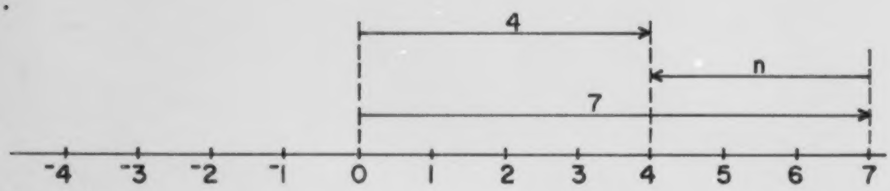
Some subtraction problems are diagrammed below. For each, answer the following.

a. What subtraction problem is diagrammed?

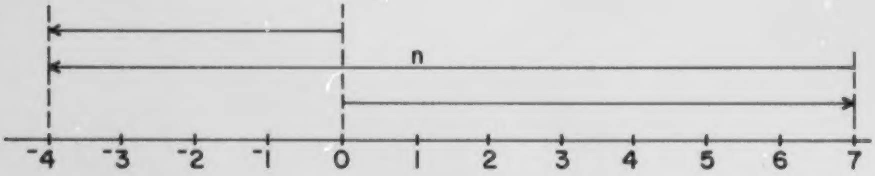
b. What is the number n?

c. What addition problem is also illustrated?

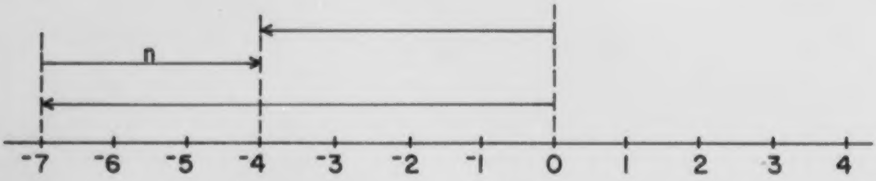
1.



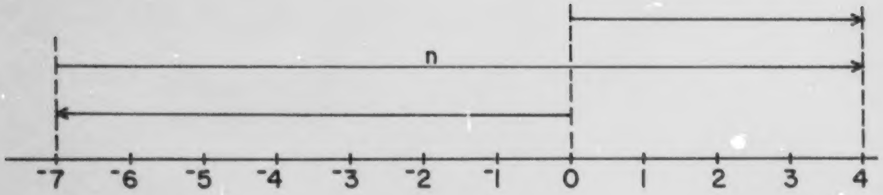
2.



3.



4.



5. Use the number line to show:

   a. $5 - 3$ and $5 + (^-3)$
   b. $5 - (^-3)$ and $5 + 3$
   c. $^-5 - 3$ and $^-5 + (^-3)$
   d. $^-5 - (^-3)$ and $^-5 + 3$

6. What conclusions can you draw from Problem 5?

In the last exercises, you may have discovered another way to think of subtraction. You found:

$$5 - 3 = 2 \text{ and } 5 + (^-3) = 2$$

So $5 - 3$ and $5 + (^-3)$ are the same numbers even though their diagrams are different.

Also, you found:

$$5 - (^-3) = 5 + 3$$

$$^-5 - 3 = {}^-5 + (^-3)$$

$$^-5 - (^-3) = {}^-5 + 3$$

In each case, subtracting a number is the same as adding its additive inverse. If $a$ and $b$ are rational numbers, then $a - b$ means $a + (^-b)$.

This property of subtraction changes every subtraction problem into an addition problem. For example:

$$3 - (^-5) = 3 + 5 = 8$$

$$7 - 4 = 7 + (^-4) = 3$$

$$^-7 - 4 = {}^-7 + (^-4) = {}^-11$$

$$1 - (^-2) = 1 + ? = ?$$

$$1 - 3 = 1 + ? = ?$$

## Exercises 17-4b

1. Use the fact that $a - b$ means $a + (^{-}b)$ to write each of the following as an addition problem. Then, do the addition.
   - a. $^{-}4 - 2$
   - b. $^{-}6 - (^{-}1)$
   - c. $8 - (^{-}3)$
   - d. $1 - 0$
   - e. $4 - 6$
   - f. $^{-}4 - 6$
   - g. $4 - (^{-}6)$
   - h. $0 - 1$

2. If you can, do the following without using the number line:
   - a. $^{-}11 - (^{-}13)$
   - b. $16 - 12$
   - c. $^{-}10 - (^{-}3)$
   - d. $8 - (^{-}2)$
   - e. $^{-}8 - 2$
   - f. $^{-}8 - (^{-}2)$
   - g. $^{-}9 - 2$
   - h. $9 - (^{-}3)$
   - i. $7 - 5$
   - j. $7 - (^{-}5)$
   - k. $2 - 9$
   - l. $2 - (^{-}9)$
   - m. $3 - 10$
   - n. $3 - (^{-}10)$

3. What is the additive inverse of:
   - a. $10$
   - b. $^{-}100$
   - c. $\frac{1}{2}$
   - d. $\frac{7}{9}$
   - e. $^{-}(\frac{8}{5})$
   - f. $^{-}(\frac{49}{51})$

4. Subtract:
   - a. $8 - 5$
   - b. $^{-}8 - 5$
   - c. $8 - (^{-}5)$
   - d. $^{-}8 - (^{-}5)$

5. Add:
   - a. $8 + (^{-}5)$
   - b. $^{-}8 + (^{-}5)$
   - c. $8 + 5$
   - d. $^{-}8 + 5$

6. Examine the numbers and the answers in Part (a) of Problems 4 and 5. Also examine Parts (b), (c), and (d). What conclusions do you draw?

7. Find x in each of these number sentences so that the statement is true:

   a. $^{-}5 + 2 = x$

   b. $x - (^{-}3) = 7$

   c. $8 + x = {}^{-}3$

   d. $x + (^{-}4) = 11$

   e. $^{-}(\frac{3}{2}) + {}^{-}(\frac{2}{3}) = x$

   f. $10 - x = {}^{-}4$

   g. $^{-}(\frac{2}{3}) + x = {}^{-}(\frac{3}{6})$

   h. $^{-}(\frac{1}{2}) - x = \frac{9}{2}$

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## 17-5. Multiplication of Rational Numbers.

You have learned how to add positive and negative numbers. Now, we want to multiply these numbers so that the familiar properties of multiplication hold.

You already know how to multiply positive numbers. For example:

$$2 \cdot 5 = 10 \qquad \frac{2}{3} \cdot \frac{5}{7} = \frac{10}{21}$$

$$3\frac{1}{2} \cdot 4 = 14$$

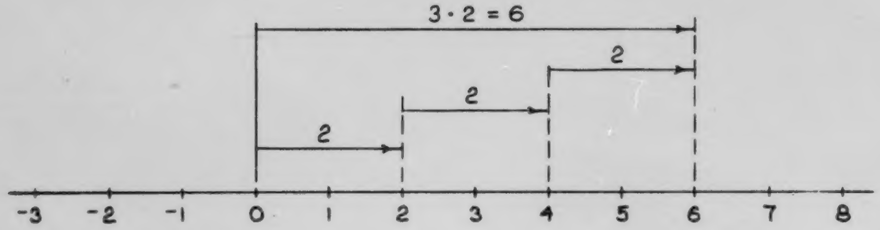
You also know that 1 is the identity element for multiplication and that 0 has a special multiplication property.

$$1 \cdot 6 = 6 \quad \text{or} \quad 1 \cdot a = a$$

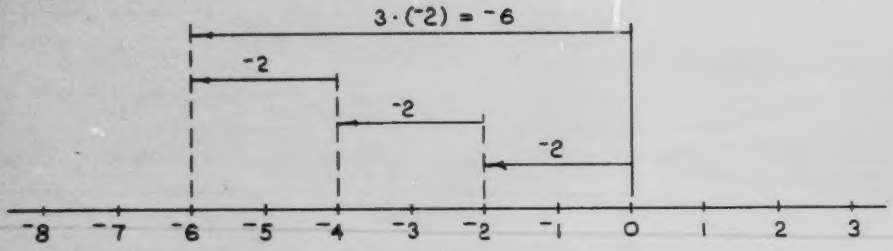
$$0 \cdot 6 = 0 \quad \text{or} \quad 0 \cdot a = 0$$

Multiplication can be shown on the number line as follows:

$$3 \cdot 2 = 2 + 2 + 2$$



The number line can be used to show the product of $3 \cdot ({}^{-}2)$ also, since this should mean $({}^{-}2) + ({}^{-}2) + ({}^{-}2)$.



## Exercises 17-5a
(Class Discussion)

1. $3 \cdot 2$ is shown above. How would you show $2 \cdot 3$?
2. How do you show $2 \cdot ({}^{-}3)$?
3. What property is illustrated by Problem 1?
4. Do $3 \cdot ({}^{-}2)$ and $2 \cdot ({}^{-}3)$ illustrate the property you named in Problem 3?
5. If the commutative property holds, use Problem 2 to find ${}^{-}3 \cdot 2$.

It is easy to use the number line to illustrate the product of positive numbers. A product like $3 \cdot ({}^{-}2)$ can also be illustrated on the number line. But what can $({}^{-}2) \cdot 3$ mean? This is more difficult and the number line does not help us. The properties of operations on numbers can help. If these new numbers are to behave like the numbers you know, then the commutative property must hold for these new numbers. For example,

$$3 \cdot ({}^{-}2) = ({}^{-}2) \cdot 3$$

The number line shows that $3 \cdot ({}^{-}2) = {}^{-}6$ so $({}^{-}2) \cdot 3$ must also equal ${}^{-}6$.

## Exercises 17-5b

1. Use the number line to multiply:
   - a. $4 \cdot 5$
   - b. $5 \cdot 4$
   - c. $3 \cdot ({}^{-}3)$
   - d. $4 \cdot ({}^{-}3)$
   - e. $2 \cdot (\frac{3}{4})$
   - f. $2 \cdot [{}^{-}(\frac{5}{2})]$
   - g. $2 \cdot (\frac{7}{3})$
   - h. $4 \cdot ({}^{-}3\frac{1}{2})$
2. Use the commutative property and the answers to Problem 1 to find:
   - a. ${}^{-}3 \cdot 3$
   - b. ${}^{-}3 \cdot 4$
   - c. ${}^{-}(\frac{5}{2}) \cdot 2$
   - d. ${}^{-}(3\frac{1}{2}) \cdot 4$
3. Complete these statements, using information from Problems 1 and 2.
   - a. The product of two positive numbers is a ________ number.
   - b. The product of a positive and a negative number is a __________ number.
4. Use the summary from Problem 3 to find:
   - a. $[2 \cdot ({}^{-}3)] \cdot 5$
   - b. $2 \cdot ({}^{-}3 \cdot 5)$
   - c. $[3 \cdot ({}^{-}5)] \cdot 7$
   - d. $3 \cdot ({}^{-}5 \cdot 7)$
   - e. $\frac{{}^{-}7}{2} \cdot [4 \cdot ({}^{-}3)]$
   - f. $(\frac{{}^{-}7}{2} \cdot 4) \cdot ({}^{-}3)$
5. What property of multiplication of rational numbers is illustrated by Problem 4?

### The Product of Two Negative Numbers

Although the number line has been used to illustrate the multiplication problems $3 \cdot 2$ and $3 \cdot ({}^{-}2)$, we find that we cannot use it for the problems $({}^{-}3) \cdot 2$ and $({}^{-}3) \cdot ({}^{-}2)$. Why?

However, we can define the product $({}^{-}3) \cdot 2$. Since we know $2 \cdot ({}^{-}3)$, we immediately know $({}^{-}3) \cdot 2$ if the commutative property of multiplication is to hold.

Now, how shall we define $(^-3) \cdot (^-2)$?
Look at part of the multiplication table.

| × | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 6 | 8 |
| 3 | 0 | 3 | 6 | 9 | ⑫ |
| 4 | 0 | 4 | 8 | 12 | 16 |

Have you used a table like this? To use it, find one factor in the left column and the other factor in the row across the top. The product is the number in the square that is in the same row as the first factor and the same column as the second factor. The product of $3 \cdot 4$ is 12, as shown in the table.

We can make a larger multiplication table by putting more numbers across the top and down the side. Then, 5, 6, 7, etc. follow naturally. There are numbers on the other side of zero, too. Let us include $^-1$, $^-2$, $^-3$, $^-4$, also. Such a table, partly filled in, is shown.

| | $^-4$ | $^-3$ | $^-2$ | $^-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $^-4$ | | | | | 0 | | | | |
| $^-3$ | | | | | 0 | | | | |
| $^-2$ | | | | | 0 | | | | |
| $^-1$ | | | | | 0 | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | | | | | 0 | 1 | 2 | 3 | 4 |
| 2 | | | | | 0 | 2 | 4 | 6 | 8 |
| 3 | | | | | 0 | 3 | 6 | 9 | 12 |
| 4 | | | | | 0 | 4 | 8 | 12 | 16 |

The products of the positive integers are given. The products of zero and all numbers shown are also given. These products are familiar to you.

From your work in the last section, you know how to fill in the top right-hand corner and the bottom left-hand corner.

### Exercises 17-5c
(Class Discussion)

1. Copy the last multiplication chart, then fill in the top right corner and the bottom left corner.
2. What number patterns are showing? The right column and the bottom row are good places to look for patterns.
3. How must the left side of the row above the zero row be filled in to keep the pattern to the right of zero?
4. What column can be filled the same way?
5. What are the patterns for the other rows and columns?
6. Fill in the rest of the table. (No fair peeking at the completed chart that follows.)

7. What is the product of $(^{-}3) \cdot (^{-}2)$?

8. It appears that the product of two negative numbers is a __________ number.

Does it seem reasonable that the pattern found in three corners of the table should be continued in the fourth? The completed chart looks like this.

| | $^{-}4$ | $^{-}3$ | $^{-}2$ | $^{-}1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $^{-}4$ | 16 | 12 | 8 | 4 | 0 | $^{-}4$ | $^{-}8$ | $^{-}12$ | $^{-}16$ |
| $^{-}3$ | 12 | 9 | 6 | 3 | 0 | $^{-}3$ | $^{-}6$ | $^{-}9$ | $^{-}12$ |
| $^{-}2$ | 8 | 6 | 4 | 2 | 0 | $^{-}2$ | $^{-}4$ | $^{-}6$ | $^{-}8$ |
| $^{-}1$ | 4 | 3 | 2 | 1 | 0 | $^{-}1$ | $^{-}2$ | $^{-}3$ | $^{-}4$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | $^{-}4$ | $^{-}3$ | $^{-}2$ | $^{-}1$ | 0 | 1 | 2 | 3 | 4 |
| 2 | $^{-}8$ | $^{-}6$ | $^{-}4$ | $^{-}2$ | 0 | 2 | 4 | 6 | 8 |
| 3 | $^{-}12$ | $^{-}9$ | $^{-}6$ | $^{-}3$ | 0 | 3 | 6 | 9 | 12 |
| 4 | $^{-}16$ | $^{-}12$ | $^{-}8$ | $^{-}4$ | 0 | 4 | 8 | 12 | 16 |

## Exercises 17-5d

1. From the completed table, state what you have discovered about the:
   a. product of two positive numbers.
   b. product of a negative number and a positive number.
   c. product of two negative numbers.

2. Find the product:
   a. $^{-}4 \cdot 0$
   b. $^{-}4 \cdot 2$
   c. $^{-}4 \cdot 5$
   d. $8 \cdot (^{-}3)$
   e. $17 \cdot (^{-}2)$
   f. $49 \cdot (^{-}5)$
   g. $^{-}6 \cdot (^{-}9)$
   h. $^{-}10 \cdot (^{-}60)$
   i. $^{-}21 \cdot (^{-}43)$
   j. $^{-}0.6 \cdot (^{-}1.4)$
   k. $4 \cdot 3 \cdot (^{-}5)$
   l. $^{-}6 \cdot 8 \cdot (^{-}12)$
   m. $^{-}3 \cdot (^{-}2) \cdot (^{-}11)$
   n. $^{-}10 \cdot (^{-}8) \cdot (\frac{^{-}2}{3})$
   o. $\frac{^{-}4}{3} \cdot (^{-}16) \cdot (\frac{^{-}3}{4})$

3. In the following problems in multiplication, put a number in the parentheses so that the statements will be correct.

   a. $(\quad) \cdot 6 = {}^{-}12$
   b. $5 \cdot (\quad) = {}^{-}15$
   c. ${}^{-}10 \cdot (\quad) = 100$
   d. ${}^{-}5 \cdot (\quad) = 20$
   e. ${}^{-}5 \cdot (\quad) = {}^{-}20$
   f. $11 \cdot (\quad) = {}^{-}110$
   g. ${}^{-}1 \cdot (\quad) = 1$
   h. ${}^{-}7 \cdot (\quad) = 0$
   i. $1 \cdot (\quad) = {}^{-}1$
   j. $6 \cdot (\quad) = {}^{-}36$
   k. ${}^{-}9 \cdot (\quad) = 81$
   l. $5 \cdot (\quad) = {}^{-}30$
   m. $(\quad) \cdot ({}^{-}10) = {}^{-}90$
   n. $(\quad) \cdot ({}^{-}50) = 100$
   o. ${}^{-}6 \cdot (\quad) = {}^{-}60$
   p. ${}^{-}(\frac{1}{2}) \cdot (\quad) = {}^{-}1$

The fact that the product of two negative numbers is positive may seem strange and unbelievable.

The above tables and the problems suggest that the product of two negative numbers is always a positive number.

We want the commutative and associative properties of addition and multiplication to hold for the rational numbers.

We also want the distributive property to hold for all rational numbers. Let us use the distributive property in the problems in the following class discussion exercises.

## Exercises 17-5e

(Class Discussion)

Complete the following statements.

1. ${}^{-}3 \cdot 0 = 0$ because we want the ________ property of zero to hold for negative numbers.

2. ${}^{-}3 \cdot [2 + ({}^{-}2)] = 0$ because the sum of ________ is zero.

3. ${}^{-}3 \cdot 2 + ({}^{-}3) \cdot ({}^{-}2) = 0$ because we want the ________ property to hold for negative numbers.

4. ${}^{-}6 + ({}^{-}3) \cdot ({}^{-}2) = 0$ because we know the product of a negative number and a positive number is a ________ number.

5. Look at what the last statement tells you about the product $^-3 \cdot (^-2)$. When it is added to $^-6$ you get 0. What is another name for the number which added to $^-6$ gives 0?

6. Therefore, $^-3 \cdot (^-2)$ must equal __?__ if the properties of non-negative numbers are to hold for negative numbers also.

The properties of multiplication that are true for the positive rational numbers are true for the whole set of rational numbers. Review these properties. They are:

The commutative property of multiplication.

$$a \cdot b = b \cdot a$$

The associative property of multiplication.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

1 is the identity element of multiplication.

$$1 \cdot a = a$$

The zero property of multiplication is true.

$$0 \cdot a = 0$$

Finally, the distributive property of multiplication over addition holds.

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

## Exercises 17-5f

1. Use the distributive property to solve the following problems.

Example: $5 \cdot [7 + (^-9)] = 5 \cdot 7 + 5 \cdot (^-9)$

$= 35 + (^-45)$

$= {}^-10$

Check: $[7 + (^-9)] = {}^-2$ and $5 \cdot (^-2) = {}^-10$

a. $^-4 \cdot (3 + 8)$

b. $^-2 \cdot (^-3 + 6)$

c. $5 \cdot [4 + (^-7)]$

d. $^-10 \cdot [^-8 + (^-1)]$

e. $6 \cdot [^-3 + (^-9)]$

f. $^-4 \cdot (^-2 + 7)$

2. Find $n$ so that these statements will be true.

a. $3 \cdot n = {^-36}$

b. $5 \cdot n = {^-75}$

c. $^-2 \cdot n = 10$

d. $^-3 \cdot n = 30$

e. $^-2 \cdot n = {^-8}$

f. $^-6 \cdot n = {^-12}$

3. Find the products.

a. $^-6 \cdot (^-10)$

b. $\frac{5}{2} \cdot 6$

c. $^-75 \cdot (^-4)$

d. $16 \cdot (^-12)$

e. $^-16 \cdot (^-12)$

f. $^-27 \cdot 0$

g. $^-16 \cdot 1$

h. $20 \cdot (^-10) \cdot (^-5)$

i. $^-2 \cdot (^-11) \cdot (^-3)$

j. $4 \cdot (^-6) \cdot 10$

k. $^-3 \cdot (^-4)$

l. $^-\frac{21}{3} \cdot (^-6)$

m. $^-\frac{23}{4} \cdot (^-4)$

n. $45 \cdot (^-13)$

o. $^-45 \cdot (^-13)$

p. $^-\frac{17}{2} \cdot 0$

q. $^-\frac{3}{4} \cdot 1$

r. $^-3 \cdot (^-15) \cdot (^-4)$

s. $^-5 \cdot 6 \cdot (^-2)$

t. $^-7 \cdot 5 \cdot 12$

---

## 17-6. Division of Rational Numbers.

Let us review the meaning of division. If you have the problem $39 \div 3$ you look for a number $n$ that, multiplied by 3, gives 39.

$$3 \cdot n = 39$$

In this case, $n$ is the rational number $\frac{39}{3}$ which is also the counting number 13. We can extend this property of division as the inverse operation of multiplication, first to negative integers, and then finally to all negative rational numbers.

You can discover how to divide rational numbers from what you know about multiplying rational numbers.

## Exercises 17-6a

(Class Discussion)

1. Below are four division problems. State the corresponding multiplication problem for each.

   a. $\frac{12}{3} = n$ $\underline{\ ?\ } \cdot n = \underline{\ ?\ }$.

   b. $\frac{^-12}{3} = n$ $\underline{\ ?\ } \cdot n = \underline{\ ?\ }$.

   c. $\frac{^-12}{^-3} = n$ $\underline{\ ?\ } \cdot n = \underline{\ ?\ }$.

   d. $\frac{12}{^-3} = n$ $\underline{\ ?\ } \cdot n = \underline{\ ?\ }$.

2. a. By inspecting the multiplication problems in Problem 1, and using what you know about multiplication, decide in each case whether n is a positive or a negative number.

   b. Copy and give a simple name for each fraction:

   $\frac{12}{3} = ?$ $\frac{^-12}{^-3} = ?$

   $\frac{^-12}{3} = ?$ $\frac{12}{^-3} = ?$

3. Fill in the words "positive" or "negative" to make statements that seem to be correct on the basis of Problem 2.

   a. The quotient of two positive numbers is a ________ number.

   b. The quotient of two negative numbers is a ________ number.

   c. The quotient of a positive number by a negative number is a __________ number.

   d. The quotient of a negative number by a positive number is a __________ number.

In multiplication and division problems involving positive and negative numbers, you may find it difficult to remember whether the answer should be positive or negative. We collect what we have learned about this situation in this one statement which may make it easier for you to remember.

When two numbers are multiplied or divided:

a) if the numbers are both positive or both negative, then the answer is positive;

b) if one number is positive and the other negative, then the answer is negative.

In the last exercises you found $\frac{^{-}12}{3}$ and $\frac{12}{^{-}3}$ are names for $^{-}4$. Since $^{-}4$ is opposite to $4$ on the number line, and $\frac{12}{3}$ is another name for $4$, $^{-}(\frac{12}{3})$ must also be the same as $^{-}4$. It is correct to write

$$^{-}(\frac{12}{3}) = \frac{^{-}12}{3} = \frac{12}{^{-}3} = {}^{-}4.$$

We choose $^{-}4$ as the common or simplest form of this rational number. In the same way,

$$^{-}(\frac{2}{3}) = \frac{^{-}2}{3} = \frac{2}{^{-}3}$$

and we choose $^{-}(\frac{2}{3})$ as the simplest form.

## Exercises 17-6b

1. Fill in the blanks so the statements will be correct.

   a. $^{-}63 \div (^{-}9) = 7$ because $7 \cdot$ ____ $= {}^{-}63$.

   b. $45 \div (^{-}5) = {}^{-}9$ because ____ $\cdot (^{-}9) = 45$.

   c. $104 \div (^{-}8) =$ ____ because $^{-}8 \cdot$ ___ $= 104$.

   d. $3 \div [^{-}(\frac{3}{2})] =$ ___ because ___ $\cdot$ ___ $=$ ____.

   e. $^{-}2 \div 3 =$ ____ because ____ $\cdot$ ____ $=$ _____.

   f. $^{-}1 \div (^{-}1) =$ ____ because ____ $\cdot$ ____ $=$ _____.

   g. $0 \div (^{-}3) =$ ____ because $^{-}3 \cdot$ ___ $=$ ____.

2. Find the products:

   a. $^{-}4 \cdot 7$

   b. $^{-}4 \cdot (^{-}3)$

   c. $2 \cdot (^{-}6)$

   d. $3 \cdot (^{-}24)$

   e. $^{-}8 \cdot (^{-}9)$

   f. $^{-}21 \cdot (^{-}35)$

   g. $\frac{^{-}3}{4} \cdot 4$

   h. $^{-}10 \cdot (\frac{^{-}2}{5})$

   i. $\frac{^{-}5}{6} \cdot {}^{-}(\frac{12}{25})$

3. Find the quotients:

a. $^{-}28 \div 7$

b. $12 \div (^{-}3)$

c. $^{-}12 \div 2$

d. $^{-}72 \div 3$

e. $72 \div (^{-}9)$

f. $735 \div (^{-}35)$

g. $^{-}3 \div 4$

h. $4 \div (\frac{^{-}2}{5})$

i. $\frac{2}{5} \div (\frac{^{-}5}{6})$

4. Find $r$ so that these statements will be true:

a. $\frac{3}{4} \cdot r = 1$

b. $\frac{^{-}3}{4} \cdot r = 1$

c. $\frac{^{-}3}{4} \cdot \frac{4}{3} = r$

d. $\frac{^{-}3}{4} \cdot r = {}^{-}1$

e. $\frac{3}{4} \cdot r = {}^{-}1$

f. $\frac{^{-}3}{4} \cdot (\frac{^{-}4}{3}) = r$

5. Reciprocals, or multiplicative inverses, are two numbers whose product is the identity element for multiplication, 1. In which parts of Problem 4 is $r$ the reciprocal of the other factor?

6. Write the set $R$ consisting of the reciprocal of each number in the set $P$:

$$P = \{6, \frac{3}{2}, 1, \frac{5}{6}, (^{-}1), (\frac{^{-}3}{4}), (\frac{^{-}7}{3})\}$$

7. Find the quotient of:

a. $\frac{^{-}18}{^{-}9}$

b. $\frac{^{-}25}{5}$

c. $\frac{^{-}30}{6}$

d. $\frac{^{-}30}{^{-}6}$

e. $\frac{30}{^{-}6}$

f. $\frac{30}{6}$

g. $\frac{^{-}100}{^{-}20}$

h. $\frac{^{-}36}{^{-}12}$

i. $\frac{64}{^{-}16}$

j. $\frac{^{-}39}{^{-}3}$

k. $\frac{0}{^{-}16}$

l. $\frac{750}{^{-}30}$

m. $\frac{^{-}432}{12}$

n. $\frac{^{-}441}{^{-}21}$

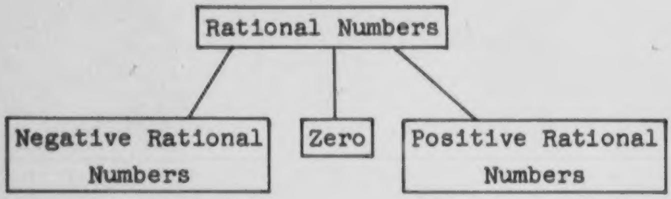
o. $\frac{^{-}484}{22}$

p. $\frac{^{-}169}{^{-}13}$

## 17-7. Summary.

In this chapter, we have extended the set of numbers to include the negative rational numbers. We now have the set of all rational numbers.

The set of rational numbers is the union of the three subsets indicated below:



Note that the number zero is neither positive nor negative.

Another subset of the set of rational numbers is the set of integers:

$$\{\ldots, {}^-4, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, 4, \ldots\}.$$

This set consists of the whole numbers and their negatives.

You have learned how to locate rational numbers on the number line. The number line has helped you to learn how to compute with negative numbers.

The *sum* of two rational numbers is:

(1) positive, when both numbers are positive.
(2) negative, when both numbers are negative.
(3) zero, when the numbers are additive inverses.
(4) positive, negative, or zero, when one number is positive and one number is negative. This is determined by the number farther from the origin.

The *difference* of two rational numbers, $a - b$, can be found by determining what number added to $b$ gives $a$. We used this principle to diagram subtraction on the number line. Later, we found that $a - b$ is the same as the sum of $a$ and the opposite of $b$.

The product or quotient of two rational numbers is:

(1) positive if both numbers are positive or both numbers are negative;

(2) negative if one number is positive and the other number is negative.

We may summarize the properties of the rational numbers as follows:

1. The rational numbers are closed under addition, multiplication, subtraction, and division (except division by zero).
2. Addition and multiplication are commutative:
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$
3. Addition and multiplication are associative:
$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
4. There is an element, 0, which is the identity for addition and an element, 1, which is the identity for multiplication.
$$a + 0 = a$$
$$a \cdot 1 = a$$
5. Every rational number has an additive inverse (opposite) and every rational number except zero has a multiplicative inverse (reciprocal).
$$a + (^-a) = 0$$
$$a \cdot \frac{1}{a} = 1$$
6. The distributive law holds.
$$a \cdot (b + c) = a \cdot b + a \cdot c$$

---

## 17-8. Chapter Review.

### Exercises 17-8

1. Add:
   - a. $7 + (^-6) + 9$
   - b. $5 + 2 + (^-11)$
   - c. $17 + (^-12) + (^-3) + (^-2)$
   - d. $7 + (^-8)$
   - e. $17 + (^-13) + 40$
   - f. $12 + 21 + (^-10) + (^-18)$

2. On January 2nd, the price of eggs was 60 cents per dozen. For the next six weeks, the price changed as shown: 1st week, up 3 cents; 2nd week, down 4 cents; 3rd week, up 7 cents; 4th week, down 8 cents; 5th week, up 6 cents; 6th week, down 7 cents. What is the price of eggs after 6 weeks?

3. A self-operated elevator is located in a store that has a 3-level parking garage below the basement. An elevator on the 1st floor makes these trips: 2 floors down; 5 floors up; 3 floors down; 1 floor down; 2 floors up; and 3 floors down. Where is the elevator?

4. Subtract:

   a. $5 - 8$
   b. $4 - 0$
   c. $0 - ({}^{-}4)$
   d. $6 - ({}^{-}6)$
   e. ${}^{-}12 - ({}^{-}15)$
   f. ${}^{-}16 - 8$

5. Denver is one mile above sea level while Pikes Peak is 14,110 feet above sea level. What is the change in altitude if you drive from the top of Pikes Peak to Denver?

6. Mt. Evans, west of Denver, is 14,264 feet above sea level. What is the change in altitude if you go from Pikes Peak to Mt. Evans?

7. Multiply:

   a. $60 \cdot 50$
   b. ${}^{-}7 \cdot 8$
   c. $9 \cdot ({}^{-}5)$
   d. ${}^{-}7 \cdot ({}^{-}9)$
   e. ${}^{-}9 \cdot 20$
   f. ${}^{-}17 \cdot ({}^{-}23)$

8. During a storm, the thermometer fell $3^{\circ}$ an hour for 6 hours.

   a. How much did the temperature change?
   b. If the temperature at the beginning of the storm was $12^{\circ}$, what was the temperature 6 hours later?

9. Divide:

a. $^{-}30 \div 6$
b. $^{-}16 \div 4$
c. $^{-}24 \div (^{-}8)$
d. $270 \div (^{-}30)$
e. $255 \div (^{-}25)$
f. $^{-}1827 \div (^{-}9)$

10. An oil well was sunk 1350 feet before oil was found. If it took 45 days to dig the well, how many feet did they average per day?

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## 17-9. Cumulative Review.

### Exercises 17-9

1. Express as a single fraction and simplify:

a. $\frac{2}{3} + \frac{2}{9}$
b. $\frac{2}{3} - \frac{2}{9}$
c. $\frac{2}{3} \times \frac{2}{9}$
d. $\frac{2}{3} \div \frac{2}{9}$
e. $(\frac{2}{3} + \frac{2}{3}) \cdot (\frac{2}{3} \div \frac{2}{3})$

2. Which of the following numbers are equal to $\frac{7}{5}$?

$1.2, \quad 1.4, \quad \frac{63}{35}, \quad \frac{91}{65}$

3. Write as decimals:

a. $\frac{3}{1000}$
b. $\frac{2}{9}$
c. $\frac{9}{2}$
d. $\frac{1000}{3}$

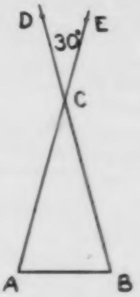
4. Find the value of the following:

a. $0.0076 \times 0.38$
b. $0.0076 \div 0.38$
c. $(7 - 0.7) + 0.07$

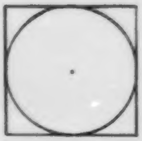
5. Fill the blanks with numerals which make correct sentences.

a. 30 in. = ____ ft.
b. 72 yd. = ____ ft.
c. 1320 ft. = ____ mi.
d. 100 mm. = ____ cm.
e. 21 m. = ____ cm.

6. The inside of a freezer is 4 feet long, 3 feet wide, and 36 inches deep. Find:
   a. its volume in cubic feet.
   b. its surface area in square feet.



7. Triangle $\triangle ABC$ is an isosceles triangle, with $m\angle A = m\angle B$. If $m\angle DCE = 30$, what is the measure of angle A?



8. A circular metal disk as large as possible is cut from a square 2 feet on a side. How much metal is wasted?

9. a. A centimeter is what percent of a meter?
   b. An inch is what percent of a yard? (to nearest whole percent.)
   c. A mile is what percent of a foot?

10. Find a number n which makes each of the following statements true.
    a. $5n = 20$
    b. $4n = 0$
    c. $5n = 2$
    d. $\frac{n}{24} = \frac{9}{8}$
    e. $\frac{n}{100} = \frac{13}{5}$

11. Find the value of the following:
    a. $^-2 + 6$
    b. $4 - (^-3)$
    c. $^-11 - (^-2)$
    d. $3 + (^-1)$

12. Find the value of the following:
    a. $6 \cdot 9$
    b. $6 \cdot (^-9)$
    c. $^-6 \cdot 9$
    d. $^-6 \cdot (^-9)$
    e. $^-2 \cdot 24$
    f. $6 \cdot (\frac{^-7}{2})$
    g. $4 \cdot (^-2\frac{1}{2})$
    h. $(\frac{^-1}{2}) \cdot (\frac{^-1}{4})$

13. Find $n$ so that these statements will be true.

a. $2 \cdot n = {}^{-}24$

b. ${}^{-}4 \cdot n = 12$

c. ${}^{-}6 \cdot n = 36$

d. ${}^{-}2 \cdot n = {}^{-}10$

14. Simplify:

a. $\frac{24}{6}$

b. $\frac{{}^{-}24}{6}$

c. $\frac{24}{{}^{-}6}$

d. $\frac{{}^{-}24}{{}^{-}6}$

e. $\frac{27}{{}^{-}3}$

f. $\frac{{}^{-}18}{3}$

g. $\frac{{}^{-}60}{{}^{-}2}$

h. $\frac{{}^{-}18}{6}$

# Chapter 18

# EQUATIONS AND INEQUALITIES

## 18-1. Sentences and Phrases.

Do you like mystery stories? Have you ever imagined yourself to be a detective like Sherlock Holmes or Nancy Drew? Sometimes, a mathematician must find one or more numbers from certain clues. In this sense he is similar to a detective.

For example, consider the following problem.

A certain number plus 3 is 8.

This English sentence may be written in mathematical language. Suppose you let x represent the "certain number" for which you are looking. Then you may write the sentence in symbols as follows:

$$\begin{array}{ccccc} \underbrace{\text{A certain number}} & \text{plus} & 3 & \text{is} & 8. \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ x & + & 3 & = & 8 \end{array}$$

This sentence in mathematical language is just one example of an equation in mathematics. It indicates that the number "$x + 3$" is another name for the number "8". In this case, it is clear that the "certain number" is 5 since 5 is the only number which when added to 3 gives the result 8. Thus, we say that 5 is the solution of the number sentence, or equation. The number "$x + 3$" is a part of this sentence. Also, the number "8" is another part of this sentence. The number "$x + 3$" and the number "8" are examples of phrases.

A phrase may represent one specific number. For example, each of the phrases

$$2 + 7, \quad \frac{8}{4}, \quad \text{II} + \text{V}, \quad 9, \quad \text{and} \quad 6 + (3 \times 4)$$

represents a number. Tell the number which each of these phrases represents.

Consider the phrase "$x + 3$". What number does the phrase "$x + 3$" represent? Does it represent 3? 5? $7\frac{1}{2}$? The answer to each of these questions depends upon the value of x.

| If "x" represents | then "x + 3" represents |
|---|---|
| $9$ | $9 + 3 = 12$ |
| $^-5$ | $(^-5) + 3 = {}^-2$ |
| $\frac{1}{2}$ | $\frac{1}{2} + 3 = 3\frac{1}{2}$ |

In order to work with phrases you must be able to translate phrases into words and also be able to translate words into phrases. A phrase may often be translated into words in several ways. For example, the phrase "x + 9" may be translated as

"the number x plus the number 9"

"9 more than the number x"

"the number x increased by 9"

### Exercises 18-1a
(Class Discussion)

1. Each of these phrases denotes a particular number. In each case, express this number in simplest form.

   a. $10 + 3$

   b. $5 - 8$

   c. $\frac{4 \times 6}{2}$

   d. $\frac{5}{10} - \frac{1}{2}$

   e. $\frac{1}{3} + \frac{3}{1}$

   f. $\frac{(^-2) + (^-19)}{3}$

   g. $7 - .5$

   h. $3\frac{1}{2} + (^-2\frac{1}{2})$

   i. $5 + (3 \times 4)$

   j. $(5 \times 3) - 4$

   k. $(\frac{1}{2} \times 4) - \frac{1}{2}$

   l. $(6 - 2) \div 1$

   m. $\frac{1}{2} + 4 - \frac{1}{2}$

   n. $(^-7) + (2 \times 3)$

   o. $(6 \times 2) + (^-15)$

2. Translate each of the following English phrases into symbols like "$x + 3$".
   - a. The sum of $x$ and 5
   - b. The product of 8 and $x$
   - c. A number $y$ added to 7
   - d. Twelve added to $z$
   - e. The sum of 6 and a number
   - f. Six subtracted from a number
   - g. The quotient obtained by dividing a number by 5
   - h. The difference between a number and 4

3. Translate each of the following phrases into words.
   - a. $x + 10$
   - b. $x - 3$
   - c. $7x$
   - d. $\frac{x}{5}$
   - e. $(^-6) + x$
   - f. $\frac{15}{x}$
   - g. $19 + x$
   - h. $17 - x$

4. Evaluate each of the phrases in Problem 3 for each of the following values.
   - a. $x = 1$
   - b. $x = {}^-5$
   - c. $x = {}^-3$
   - d. $x = 0$
   - e. $x = \frac{1}{5}$
   - f. $x = {}^-15$
   - g. $x = 6$
   - h. $x = 12$
   - i. $x = 5$
   - j. $x = 9$

## Exercises 18-1b

1. Translate each of the following English phrases into symbols.
   - a. Four less than $x$
   - b. The number $x$ added to 7
   - c. The number $x$ subtracted from 30
   - d. The product of 15 and $x$
   - e. One-fourth of the number $x$
   - f. The number $x$ plus 10
   - g. The number 7 multiplied by $x$
   - h. The number obtained by subtracting 11 from $x$
   - i. The number $x$ divided by 2
   - j. The difference between $x$ and 6
   - k. The number $x$ minus 9

2. For each one of the phrases in Problem 1, find the number represented by the phrase if $x = 12$.

3. Translate each of the following phrases into words.
   a. $x + 1$
   b. $x - 3$
   c. $2x$
   d. $15 + x$
   e. $\frac{18}{x}$
   f. $20 - x$
   g. $6x$
   h. $^{-}4 + x$

4. Find the number represented by each of the open phrases in Problem 3 if $x = 6$.

5. Find the number represented by each of the open phrases in Problem 3 if $x = {}^{-}2$.

6. A number need not always be represented by $x$. Translate each of the following phrases into symbols using the letter label of each part as the number. For example, in Part (a) use "a" as the number.
   a. The sum of six and a number
   b. Eight times a number
   c. Eight times a number added to 1
   d. Three subtracted from 8 times a number
   e. Seven times a number divided by 4
   f. Three more than twice a number
   g. The product of 5 and the sum of a number and 2
   h. Ten minus seven times a number
   i. Twelve divided by the sum of a number and 1
   j. The product of two factors, one of which is the sum of three and a certain number and the other of which is the sum of 4 and the same number
   k. The difference between 25 and the product of 3 and a number
   l. The quotient obtained by dividing a number by the sum of 2 and the same number

7. Find the number represented by each of the phrases in Problem 6 if the number is $^{-}3$.

8. Translate each of the following phrases into words. In each case, the letter represents a number.

   a. $m - 6$

   b. $4n$

   c. $2x + 5$

   d. $5 - d$

   e. $2w + 15$

   f. $15 + 2d$

   g. $9 - 3d$

   h. $\frac{x + 12}{2}$

   i. $7(x + 1)$

   j. $5 - 2n$

9. Find the number represented by $2n + 5$ for each of the following values of $n$.

   a. $n = 5$

   b. $n = {}^{-}5$

   c. $n = 0$

   d. $n = 1$

   e. $n = {}^{-}1$

   f. $n = {}^{-}(\frac{1}{2})$

   g. $n = \frac{1}{2}$

   h. $n = {}^{-}2$

   i. $n = \frac{3}{2}$

   j. $n = {}^{-}3$

   k. $n = 5\frac{1}{2}$

   l. $n = {}^{-}2\frac{1}{2}$

10. Find the number represented by $6 - 3q$ for each of the following values of $q$.

   a. $q = 0$

   b. $q = 1$

   c. $q = {}^{-}1$

   d. $q = 5$

   e. $q = \frac{1}{3}$

   f. $q = {}^{-}(\frac{2}{3})$

   g. $q = \frac{1}{2}$

   h. $q = {}^{-}(\frac{3}{2})$

---

## 18-2. Sentences and Their Solutions.

We use sentences every day in talking, reading, and writing. In mathematics, we also find it necessary to use sentences. A sentence about numbers is often written in the form of an equation such as

$$x + 7 = 9.$$

This number sentence says,

> "If seven is added to a certain number, x, the result is nine."

The following are examples of some other sentences:

"The sum of 8 and 7 is 15".

"$x + 3 = 8$".

"$4 + 5 = 3 \cdot 3$".

"$3 < 2 + 4$".

"$2^2 > 2$".

The above examples are sentences because each of them consists of two phrases connected by a verb. What are the verbs in these sentences? The first one is easy to find. The word "is" is the verb. What are the symbols for verbs in the remaining sentences? Your answers should be "$=$", "$=$", "$<$", and "$>$". The three most common verbs in number sentences are "$=$", "$<$", and "$>$".

A number sentence which uses the verb "$=$" is called an *equation*. A number sentence which uses the verb "$<$" or "$>$" is called an *inequality*.

To solve the equation "$x + 2 = 9$" means to find the value of $x$ which makes the sentence true. In this case, the value $x = 7$ is the only number which, when added to 2, gives the result 9.

### Exercises 18-2a

1. Solve the following equations.

a. $x + 3 = 5$

b. $y + 5 = 12$

c. $k + 13 = 15$

d. $z + 2 = 6$

e. $p + 5 = 11$

f. $t + 25 = 31$

g. $a + 17 = 42$

h. $m + 10 = 5$

2. In each case, find the value of the letter which makes the sentence true.

a. $x - 7 = 2$

b. $y - 5 = 5$

c. $n - 9 = 2$

d. $a - 3 = 7$

e. $x - 3 = 6$

f. $p - 15 = {}^{-}1$

g. $x - 5 = 3$

h. $y - 12 = {}^{-}4$

3. Solve the following equations.

   a. $4b = 12$
   b. $3a = 15$
   c. $5w = 35$
   d. $6d = 72$
   e. $9m = 54$
   f. $13x = {}^{-}13$
   g. $7y = {}^{-}56$
   h. $3x = 8\frac{1}{4}$

4. Solve the following equations.

   a. $\frac{n}{3} = 2$
   b. $\frac{a}{4} = 4$
   c. $\frac{k}{8} = {}^{-}2$
   d. $\frac{x}{5} = 7$
   e. $\frac{d}{9} = 2$
   f. $\frac{h}{-3} = 5$
   g. $\frac{s}{-2} = {}^{-}7$
   h. $\frac{y}{8} = 4$

5. In each case, find the value of the letter which makes the sentence true.

   a. $x + 1 = 5$
   b. $y + 3 = 0$
   c. $\frac{a}{2} = 9$
   d. $5x = 20$
   e. $n - 15 = {}^{-}2$
   f. $\frac{x}{4} = {}^{-}8$
   g. $7y = 49$
   h. $8g = {}^{-}32$
   i. $14 - x = 12$
   j. $\frac{m}{3} = {}^{-}3$
   k. $\frac{m}{2} = 0$
   l. $n - 6 = {}^{-}6$
   m. $15n = 225$
   n. ${}^{-}11 + x = {}^{-}8$
   o. $x + 1 = x + 1$
   p. $x + 1 = 1$
   q. $\frac{p}{5} = 6\frac{1}{4}$
   r. ${}^{-}5 + y = {}^{-}9$

The statement

$$x + 3 = 7$$

tells us something about the number $x$. We might say that it gives us a clue about $x$. We see that 4 is the only number which, when added to 3, gives 7. Thus, $x$ must be 4. In this case, our clue was so strong that it completely identified the number $x$. We say that 4 is the solution of this equation or sentence.

Here is another sentence

$$y + 3 > 7$$

This sentence gives us a clue about a number $y$. But this time our clue is not so strong as to identify completely the number $y$. You can see that $y$ might be any of the numbers 5, 6, 12. In fact, $y$ might be any number greater than 4, but $y$ could not be 4 or a number less than 4. We can see then that, although this time our clue does not tell us exactly what the number $y$ is, it does narrow down the list of suspects. In this sentence, the letter $y$ may be replaced by many different numbers to obtain a true statement. The set of all such numbers is called the solution set of the inequality or sentence.

Consider the inequality $x - 4 > 7$. How large must the number $x$ be in order to make this sentence true? Is $5 - 4 > 7$? Is $12 - 4 > 7$? Do you see that "$x - 4 > 7$" is a true sentence if $x$ is any number greater than 11? Also, "$x - 4 > 7$" is a false sentence for any other value of $x$. The solution of this inequality, then, is the set of all numbers which are greater than 11.

## Exercises 18-2b

1. Translate each of the following sentences into symbols.
    a. If the number $x$ is added to 5, the result is equal to 13.
    b. If 3 is subtracted from $x$, the result is equal to 7.
    c. The product of 8 and $x$ is equal to 24.
    d. When $x$ is divided by 4, the result is 9.
    e. When ten is added to the number $x$, the sum is 21.
    f. If the number 7 is multiplied by $x$, the product is $^-35$.
    g. If the number 11 is subtracted from $x$, the difference is $^-5$.
    h. The number $x$ minus 6 is 15.
    i. The number $x$ divided by 2 is equal to $^-7$.
    j. Six more than twice a number is 4.

2. Solve each of the equations in Problem 1.

3. Translate each one of the following sentences into symbols.
   a. The sum of $x$ and 2 is greater than 4.
   b. The product of 5 and $x$ is less than 10.
   c. The result of dividing $x$ by 7 is greater than 2.
   d. The number $x$ minus three is greater than 6.
   e. Five subtracted from $x$ is less than 13.
   f. The product of 3 and the number $x$ is greater than $^{-}9$.
   g. Two less than three times a number is greater than 7.

4. For each one of the inequalities you wrote in Problem 3 use your knowledge of arithmetic to find the solution set.

5. Translate each one of the following sentences into words. Use the term "a number" or "a certain number" to represent the unknown number.
   a. $y + 2 = 5$
   b. $z + (^{-}3) = 7$
   c. $2a = {}^{-}10$
   d. $h - 5 > 9$
   e. $5m < 15$
   f. $7 - k = 2$
   g. $d - 3 < 4$
   h. $\frac{w}{3} > 9$
   i. $k - 7 = {}^{-}2$
   j. $\frac{c}{^{-}30} = 6$

6. Solve each equation and inequality in Problem 5.

7. Write three other sentences which will have the same solution set as the sentence $y = 5$.

---

## 18-3. Formulas.

You have already used a special kind of equation. For example, to find the number of square units of area in a rectangle you used the following:

$$A = \ell w$$

This is an abbreviation of a rule. In words, this rule is

> "The number of square units of area in a rectangle is (or, is equal to) the product of the number of units in the length and the number of like units in the width."

When such a rule is abbreviated and written in the form of an equation, it is called a formula. If the length and width of a rectangle are known, then this formula may be used to find the area of that rectangle. The same formula is used to find the area of all rectangles.

Many products are packed in cylindrical cans. Usually, a label made of paper is pasted on the outside surface of the can. This label covers all of the can except the circular top and the circular bottom. The portion of the can covered by the label is called the lateral surface of the can. The formula for the area of the lateral surface of a cylinder is

$$A = 2\pi rh$$

where

$A$ = the number of units in the lateral area of the cylinder

$r$ = the number of units in the radius of the base of the cylinder

$h$ = the number of units in the height of the cylinder

Example: Find the area of a label pasted on a cylindrical can which has a base radius of $3\frac{1}{2}$ inches and a height of 12 inches. (Let $\pi = \frac{22}{7}$.)

In the formula $A = 2\pi rh$

$$r = 3\tfrac{1}{2}, \quad \text{or} \quad \frac{7}{2}$$

$$h = 12$$

$$A = 2 \cdot \frac{22}{7} \cdot \frac{7}{2} \cdot 12$$

$$A = 264$$

The area of the label is approximately 264 square inches.

## Exercises 18-3

1. The formula for finding the perimeter of a rectangle is $p = 2\ell + 2w$.
   a. Find the perimeter of a rectangle whose length is 7 feet and whose width is 4 feet.
   b. Find the perimeter of a rectangle whose length is $8\frac{1}{2}$ inches and whose width is $5\frac{1}{4}$ inches.
2. The formula for finding the area of a triangle is $A = \frac{1}{2}bh$.
   a. Find the area of a triangle whose base (b) is 14 inches and whose height (h) is 7 inches.
   b. Find the area of a triangle whose base is 8 inches and whose height is $6\frac{1}{2}$ inches.
3. The formula for the area of a square is $A = s^2$.
   a. Find the area of a square whose side is 15 inches.
   b. Find the area of a square whose side is 7.4 inches.
4. Doug mows a lawn 60 feet long and 40 feet wide. He mows another lawn which is 65 feet long and 36 feet wide. Use the formula $A = \ell w$ to find the area of each lawn. Find the difference in the areas of these two lawns.
5. Use the formula $p = 2\ell + 2w$ to find the perimeter of a rectangular picture frame which is $6\frac{1}{2}$ feet long and 4 feet wide.
6. Use the formula $A = s^2$ to find the area of a square rug whose side is $12\frac{1}{2}$ feet.
7. The formula for the area (A) of a parallelogram is $A = bh$ where b is the length of the base and h is the height. Find the area of a parallelogram if
   a. $b = 6.4''$ and $h = 7.5''$
   b. $b = 8\frac{1}{2}''$ and $h = 3\frac{1}{4}''$
   c. $b = 18.2''$ and $h = 9.6''$

8. The formula used in finding simple interest is written $i = prt$, where

   $i$ is the number of dollars of interest
   $p$ is the number of dollars in the principal (or amount borrowed)
   $r$ is the rate (or percent) of interest per year
   $t$ is the number of years

   a. Find the simple interest for a bank loan of $750 for 3 years at 6% interest.
   b. Find the simple interest for a business loan of $1240 for $2\frac{1}{2}$ years at 5% interest.
   c. Find the simple interest for a business loan of $3600 for 1 year and 9 months at 6% interest.

9. The formula for finding the circumference of a circle is $C = 2\pi r$. (Use 3.14 for $\pi$).

   a. Find the circumference of a circle whose radius is 10 inches.
   b. Find the circumference of a wheel whose diameter is 26 inches.
   c. Find the circumference of a tabletop whose diameter is 5 feet.
   d. Find the circumference of a batter's circle whose diameter is 5.6 feet.

10. A softball diamond is a square 60 feet on a side.

    a. Using the proper formula, find the perimeter of the diamond.
    b. Using the proper formula, find the area of the softball diamond.

11. The formula shown at the right may be used to find the total distance traveled if the rate of travel does not change. For example, if the rate is 25 miles per hour on a trip, it remains at 25 miles per hour throughout the trip.

$d = rt$

$d$ = number of miles in the distance (in this example)

$r$ = rate of travel in miles per hour

$t$ = number of units of time

a. Find the distance traveled by an automobile moving at a rate of 45 miles per hour (m.p.h.) for 13 hours.

b. Find the distance traveled by a plane at a speed of 420 m.p.h. for $3\frac{3}{4}$ hours.

c. Find the distance between Benton and Cedarville if it takes a train $4\frac{1}{4}$ hours to cover this distance at a speed of 56 m.p.h.

12. Find the areas of the following circles. (Use 3.14 for $\pi$)

a. whose radius is 13.

b. whose diameter is 22.

c. whose radius is 7.5.

13. Which is larger in area: a square whose side is 4 feet or a circle whose radius is 2 feet? How much larger is it? (Use $\pi = 3.14$).

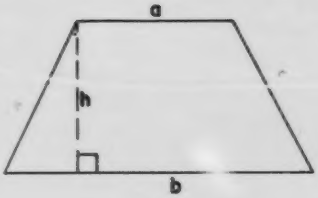
14. The area of the figure at the right is given by the formula

$$A = \frac{1}{2}h(a + b)$$

where

$h$ is the measure of the height or altitude

$a$ and $b$ are the measures of the bases.



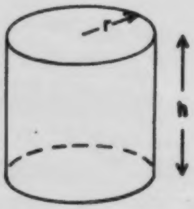
Find the area of this figure if

a. $h = 12$, $a = 7$, $b = 11$

b. $a = 6$, $b = 15$, $h = 7$

c. $b = 18$, $h = 11$, $a = 15$

15. a. Find the volume of the cylindrical tank pictured at the right if the radius is 1 foot and the height is $3\frac{1}{2}$ feet. (Use $\frac{22}{7}$ for $\pi$ ).



$V = \pi r^2 h$

b. Find the capacity of the tank in gallons. One cubic foot holds $7\frac{1}{2}$ gallons.

16. The formula $F = \frac{9}{5}C + 32$ may be used to convert a temperature reading on a Centigrade thermometer to a temperature reading on a Fahrenheit thermometer. Find the correct Fahrenheit temperature reading for each of the following readings on a Centigrade thermometer.

a. $10^\circ$ b. $100^\circ$ c. $35^\circ$ d. $0^\circ$ e. $47^\circ$

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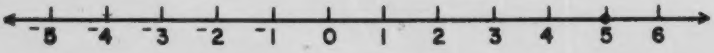
## 18-4. Graphing Solution Sets of Sentences.

We can draw a picture of the solution set of a sentence on the number line. This picture is called the graph of the sentence.

**Example 1**: Consider the sentence

$$x + 3 = 8$$

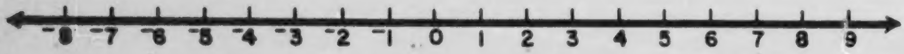
The solution set of this equation has only one member. The solution set is $\{5\}$. On the number line this set can be represented by a solid dot at the point 5.



**Example 2**: Consider the sentence

$$x + 3 = 3 + x$$

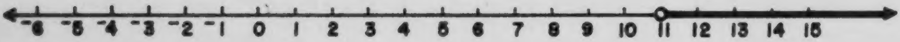
First find the solution set. Try $x = 6$. Does $6 + 3 = 3 + 6$? Try 13. Does $13 + 3 = 3 + 13$? Try any other number. What property of addition tells us that this sentence is true no matter what number we use for $x$? Thus, the solution set for this equation is the set of all numbers. The graph of this solution set is a heavy dark line as shown below.



**Example 3**: Consider the inequality

$$x - 4 > 7$$

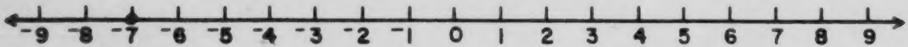
Earlier in this chapter, we found that the solution set of this inequality is the set of all numbers which are greater than 11. The graph of this solution set is shown below. The "open" circle ( ○ ) shows that the number 11 is not included in the graph.



**Example 4**: Consider the equation

$$11 + x = 4$$

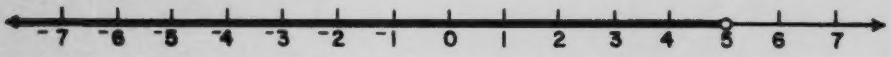
What is the solution set? Try some numbers. Recall what you have learned about negative numbers. What is $11 + (^-7)$? Is $^-7$ a member of the solution set? Can you find any other members of the set? You should not be able to do so. The solution set for this equation is $\{^-7\}$. This is shown on the number line below.



**Example 5**: Consider the inequality

$$x - 4 < 1$$

For what numbers is this inequality a true sentence? Try some numbers. Do you find that the solution set of this inequality is the set of all numbers less than 5? On the number line below this is represented as an "open" circle at the point corresponding to 5 and a heavy black line drawn along all points of the number line which lie to the left of 5.



Some solution sets contain only one member. The graph of such sets may be represented by a single filled circular mark on the number line. The circle is drawn at the point which corresponds to the number in the solution set.

If the solution set is the set of all numbers, the graph is a heavy, dark line along the entire number line. In this case, the solution set is represented by the entire number line.

The graph of the solution set for inequalities is often represented by part of a number line. The inequalities considered in this chapter were all represented by half-lines on the number line. An open, or empty, circle was used to indicate a point not included in the graph.

## Exercises 18-4

1. Find the solution set for each of the following sentences.
   - a. $x + 2 = 6$
   - b. $4 + x = 0$
   - c. $2x = 6$
   - d. $3x < 3$
   - e. $x - 4 > 1$
   - f. $\frac{x}{5} = {}^-1$
   - g. $2x < 10$
   - h. $4 - x > 1$

2. For each one of the sentences in Problem 1 graph the solution set on a number line.

3. Find the solution set for each of the following sentences.

| | | | |
|---|---|---|---|
| a. | $x + 1 = 1 + x$ | e. | $3w = {}^-15$ |
| b. | $y - 1 > 0$ | f. | $14 + x = 13$ |
| c. | $1 - b > 0$ | g. | $13 - x = 14$ |
| d. | $a + 2 = 1 + a$ | *h. | $\frac{2}{x} = {}^-1$ |

4. For each one of the sentences in Problem 3 graph the solution set on a number line.

5. Find the solution set for each of the following sentences.

| | | | |
|---|---|---|---|
| a. | $2x + 3 = {}^-7$ | f. | $2n + 1 = 1 + 2n$ |
| b. | $3m - 1 = 5$ | g. | $6x + 7 = 43$ |
| c. | $2b + 1 > 9$ | h. | $4m - 5 < 19$ |
| d. | $3x + 3 < 0$ | i. | $\frac{y}{3} + 1 = 0$ |
| e. | $5m - 1 = {}^-16$ | j. | $\frac{y}{4} - 2 > 0$ |

6. For each one of the sentences in Problem 5 graph the solution set on a number line.

For each of Problems 7-14,

a. Write a sentence.
b. Find the solution set of the sentence.
c. Graph the solution set of the sentence.

7. A certain number added to seven equals fourteen.

8. If six is added to two times a certain number the result is 10.

9. The product of 5 and a certain number is greater than zero.

10. If 3 is subtracted from a certain number the result is less than 1.

11. A certain number is multiplied by three. Then 2 is added to this new number. This result is greater than 8.

12. The sum of a certain number and two times that number is 12.

13. The sum of a certain number and itself is less than 4.

14. If 1 is added to four times Susie's age in years, the result is 21.

## 18-5. Summary.

In your work in this chapter you learned how to translate English sentences into equations and inequalities. These skills will be very helpful in your future work in mathematics. Also, you found solution sets for sentences and made graphs of these solution sets. The technique of graphing a solution set is very common and useful in many fields of mathematics.

Since vocabulary is essential in our work, the important terms of this chapter are listed below for your review and ready reference.

1. PHRASE. A phrase is a part of a sentence which describes or represents a number. Some phrases represent specific numbers such as $15$ or $8 - 3$. Other phrases such as $11x$ or $y + 2$ do not represent a specific number.

2. SENTENCE. A sentence consists of two phrases connected by a verb. The most common verbs in mathematics are "=", "<", and ">".

3. EQUATION. An equation is a sentence which uses the verb "=". Equations are used to express the idea that two phrases are names for the same number.

4. INEQUALITY. An inequality is a sentence which uses the verb "<" or the verb ">". Inequalities are used to express the idea that two numbers are not equal.

5. SOLUTION SET. A solution set is the set of values which make a sentence true. When we find the solution set of a sentence, we say we have solved the equation or the inequality.

6. **FORMULA.** A formula is a special kind of equation. A formula is a mathematical abbreviation of a rule.

7. **GRAPH.** A graph of a sentence is a picture representation of the solution set of the sentence.

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## 18-6. Chapter Review.

### Exercises 18-6

1. Translate each of the following phrases into symbols.
   a. The sum of some number and forty-three.
   b. The product of some number and eleven.
   c. Six less than three times some number.
2. Name the three most common "verbs" used in sentences.
3. Translate each of the following phrases into words.
   a. $2x + 9$
   b. $x - ({}^{-}2)$
   c. $\frac{12}{3x}$
4. Find the number represented by each of the open phrases in Problem 3 when $x = 3$.
5. How can you recognize an equation?
6. Translate each of the following sentences into symbols.
   a. The number $y$ added to sixteen is twelve.
   b. The sum of $x$ and 14 is greater than 15.
   c. Six less than three times a certain number is less than 12.
   d. The product of 4 and a certain number is 20.
7. What is a solution set?
8. Find the solution set of each of the sentences in Problem 6.
9. Write two other sentences which have the same solution set as $y > 2$.

10. Use the formula $p = 2\ell + 2w$ to find the perimeter of a rectangle whose length is $5\frac{1}{2}$ feet and whose width is $2\frac{2}{3}$ feet.

11. Use the formula $A = s^2$ to find the area of a square flower garden whose side is $6\frac{1}{4}$ feet.

12. What is a graph of a sentence?

13. Graph the solution set of each of the parts of Problem 6.

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## 18-7. Cumulative Review.

### Exercises 18-7

1. The average of five numbers is obtained by adding the numbers and dividing the sum by five. Find the average of the numbers $\{^-8, 10, 0, ^-1, ^-3\}$.

2. Perform the indicated operations and simplify:
   - a. $3\frac{1}{3} + 1\frac{1}{2}$
   - b. $10 \times 1\frac{3}{5}$
   - c. $7\frac{3}{16} - 1\frac{3}{4}$
   - d. $3\frac{1}{2} \div 10$
   - e. $10 \div \frac{4}{15}$
   - f. $33 \div 5\frac{1}{2}$

3. Which of the following numbers is equal to $\frac{3}{5}$?

   0.60, 0.6, 6%, 0.06, 60%

4. Round each of the following as indicated:
   - a. 987.654 (to the nearest hundred)
   - b. 987.654 (to the nearest hundredth)

5. Do the following operations.
   - a. $15 + 0.012 - 1.3$
   - b. $17.045 \div 100$
   - c. $17.045 \times 100$
   - d. $31 \div 15.1$ (round answer to the nearest tenth)

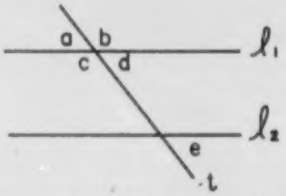
6. Multiply 3 by $^-6$, then subtract $^-12$ from the product, then divide the result by $^-6$.

7. Find the following products:
   a. $^-10 \times {}^-10 \times 0$
   b. $1 \times {}^-3 \times {}^-41$
   c. $^-8 \times 5 \times {}^-8$
   d. $0.5 \times {}^-0.2 \times 8$

8. If the length of a segment is correctly stated to be $7\frac{1}{4}$ inches, it might be as short as ______, or as long as ____.

9. The area of a square is 64 sq. units.
   a. What is its length?
   b. What is its width?
   c. What is its perimeter?

10. If $\ell_1$ and $\ell_2$ are parallel lines cut by the transversal t,



   a. Name an angle which corresponds to ∠e.
   b. Name an angle which is supplementary to ∠b.
   c. Name an angle which is vertical to ∠b.
   d. If m(∠a) = 15, find m(∠b), m(∠c), m(∠d), m(∠e).

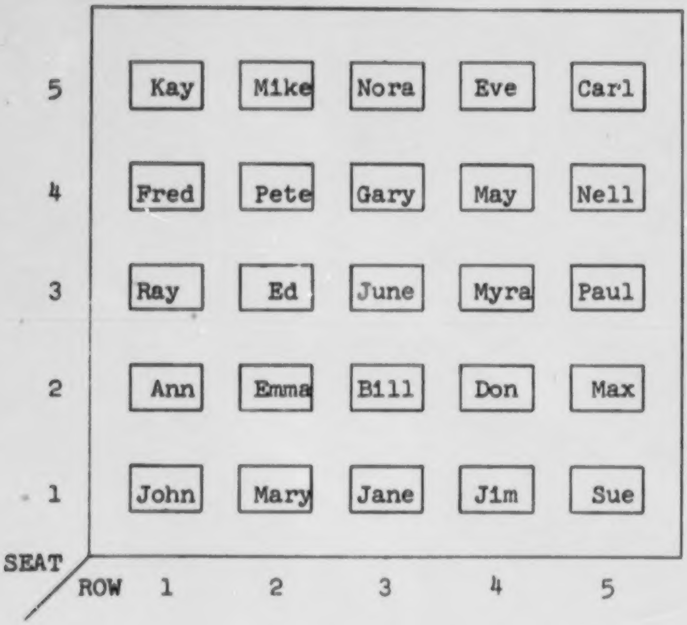
# Chapter 19

# COORDINATES IN THE PLANE

## 19-1. Locating Points in a Plane.

We have seen that the number line gives us a way of using numbers to express the location of points on a line. We can also use numbers to locate points in a plane. But now we need to use two numbers instead of one to locate each point. Let us see some examples of how this may be done.

Here is a picture of a classroom.



| | | | | | |
|---|---|---|---|---|---|
| 5 | Kay | Mike | Nora | Eve | Carl |
| 4 | Fred | Pete | Gary | May | Nell |
| 3 | Ray | Ed | June | Myra | Paul |
| 2 | Ann | Emma | Bill | Don | Max |
| 1 | John | Mary | Jane | Jim | Sue |
| SEAT / ROW | 1 | 2 | 3 | 4 | 5 |

We can think of the classroom as part of a plane and the seats as points in the plane. We see that Ed sits in Row 2, Seat 3. We could shorten this to (R 2, S 3). We could even shorten it to just (2, 3) if we are careful to remember to write the row number first and the seat number second. Now you see a way in which we can use a pair of numbers to locate a position in a plane.

We just saw that Ed's position is (2, 3). Do you see that Bill's position is (3, 2)? Do you see why it is important to keep the order decided upon and to write the row number first and the seat number second?

## Exercises 19-1a
(Class Discussion)

1. a. Who sits at (4,5)?
   b. Who sits at (5,4)?
   c. What pair of numbers gives Pete's position?
   d. What pair of numbers gives Don's position?
2. Write the set of all the number pairs which correspond to seats occupied by girls.
3. Consider the set of students:
   {Fred, Pete, Gary, May, Nell}.
   a. What do their positions have in common?
   b. Write the set of number pairs corresponding to these students.
   c. What do these number pairs have in common?
4. Consider the set of students:
   {Jane, Bill, June, Gary, Nora}.
   a. What do their positions have in common?
   b. Write the set of number pairs corresponding to these students.
   c. What do these number pairs have in common?
5. a. What is the intersection of the sets of students in Problems 3 and 4?
   b. What is the intersection of the sets of number pairs in Problems 3 and 4?
6. a. Find the set of students having row number and seat number equal.
   b. Find a geometric way of describing the positions of these students.

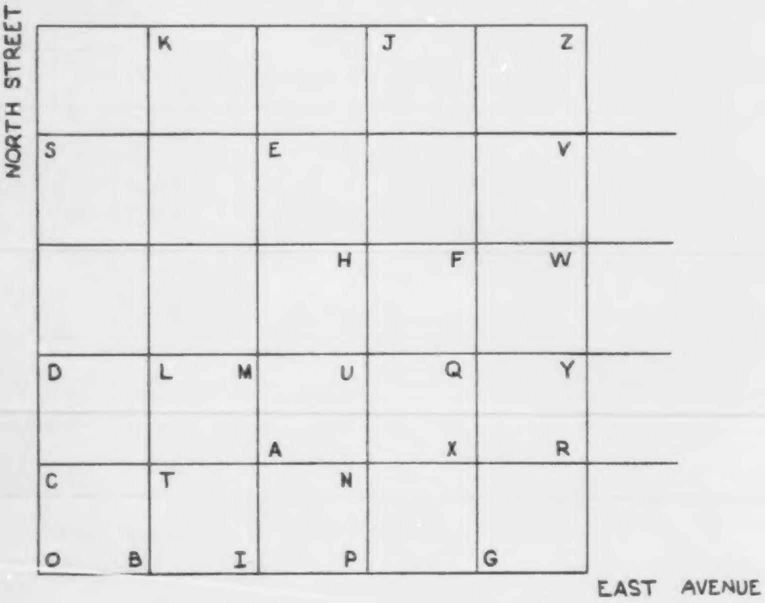
7. a. Find the set of students having their seat number greater than their row number.

   b. How can you express their position geometrically?

Here is another example to illustrate the idea of using pairs of numbers to locate points in a plane.

Example: Shown below is a map of the town of Walnut Falls. (This town was selected because all its streets run North-South or East-West and all its blocks are of the same length.) The following sites have been marked with the first letter in their names:

Auditorium
Bank
City Hall
Drug Store
Elementary School
Fire House
Gas Station
Hospital
Insurance Office
Jail
Kindergarten
Library
Museum
Newsstand
Office Building
Park
Quadrangle
Restaurant
Supermarket
Telephone Company
University
Veterinarian
Wharf
X-ray Laboratory
Yacht Club
Zoo



You are a policeman standing in front of the office building at East Avenue and North Street. A stranger with a sick dog asks you where the veterinarian is located. What can you tell him? You can tell him that the veterinarian is located 5 blocks East and 4 blocks North. This can be shortened to (5 E, 4 N).

## Exercises 19-1b
(Class Discussion)

1. You are still a policeman at the corner of East Avenue and North Street. Expressing locations as in (5 E, 4 N), how can you tell a stranger the location of:
    a. the Kindergarten?
    b. the Telephone Company?
    c. the Wharf?
    d. the Yacht Club?
    e. the Quadrangle?
2. How can you give the location to a stranger if he wants to
    a. eat lunch?
    b. see a kangaroo?
    c. hear a concert?
    d. see the collection of local Indian relics?
    e. read a book?
    f. buy a newspaper?
    g. have his tonsils taken out?
    h. take a course in advanced calculus?
3. In giving the location of the Fire House which was (4 E, 3 N) we can drop the letters E and N and write simply (4,3). If we do this, we must agree always to write the "East address" first and the "North address" second.

$$\underset{\text{East address}}{\searrow}(4,\ 3)\underset{\text{North address}}{\swarrow}$$

Give the location of each of the points in Problem 2 in this way.

4. If a stranger asks you for the location of the park, you probably will say, "three blocks East". If he then asks how many blocks North it is, you may say, "no blocks North" or "zero blocks North". Then you may express this location in the same way as in Problem 3 as (3, 0) which means 3 blocks East, 0 blocks North. Now to what location will you send a person who wants to:
   a. buy a jar of peanut butter?
   b. pay his taxes?
   c. get a tire repaired?
   d. enroll his eight year old boy in school?
   e. take out a fire insurance policy?
   f. report a fire?
   g. have a burn bandaged?
   h. get some ointment for a burn?
   i. borrow a book?
   j. rent an office?

5. a. What buildings are at corners having an East address and a North address which are the same?
   b. On what kind of geometrical figure do these corners lie?

6. a. What buildings have their North address greater than their East address?
   b. Where do these buildings lie?

7. a. List all the other locations which lie on the same street as the Restaurant and the Wharf.
   b. List the corresponding number pairs.
   c. What do these number pairs have in common?
   d. Do you see that buildings lying on the same North-South street all have the same East address?

8. a. List all the other buildings which are on the same street as the Drug Store and the Quadrangle.
   b. List the number pairs corresponding to these buildings.
   c. What do these number pairs have in common?
   d. Do you see that buildings lying on the same East-West street have the same North address?

9. Suppose you wish to walk from the Office Building to the Fire House.
    a. How many blocks East will you walk?
    b. How many blocks North?
    c. How many blocks in all?
10. In Problem 9 suppose it were possible to cut straight across the blocks, how far would you have to walk? (Give the answer in blocks, not in inches. To do this you will have to make your own ruler by marking the edge of a piece of paper in blocks. Measure between the corners at which the buildings are located, not between the buildings themselves.)

## 19-2. Coordinates in the Plane.

Now that you have answered the questions in the last section, you are ready to take up the general problem of locating points in a plane.

First draw two number lines, one horizontal and one vertical.

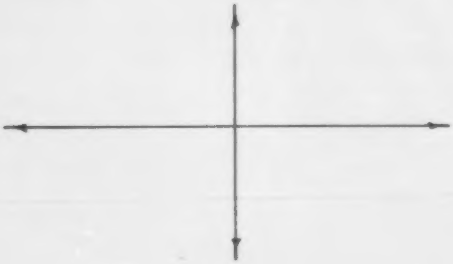


Figure 1

Notice that the four angles determined by these lines are all right angles. Now place coordinates on each line using these rules:

1. The point of intersection of the two lines is to be the zero-point (origin) on both lines.

2. The same unit of distance is to be used on both lines.

Now you have a picture that looks like this:

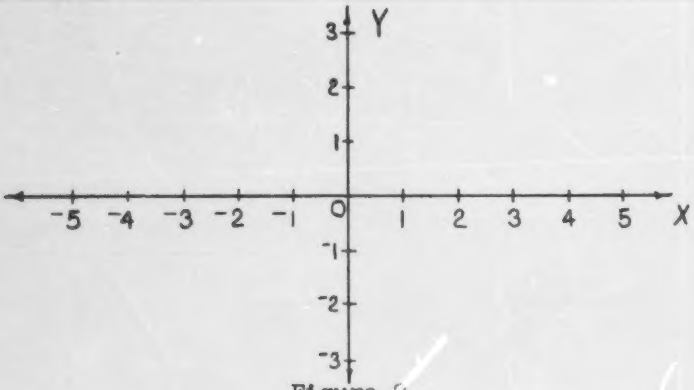

Y
3
2
1
⁻5
⁻4
⁻3
⁻2
⁻1
0
1
2
3
4
5
X
⁻1
⁻2
⁻3

Figure 2

The horizontal line is called the X-axis and the vertical line is called the Y-axis. The plural of the word "axis" is "axes" (pronounced axe-ease, with the accent on the axe). The point of intersection of the axes is called the origin.

Now pick a point S in the plane of these two axes. In Figure 3, a point S has been marked.

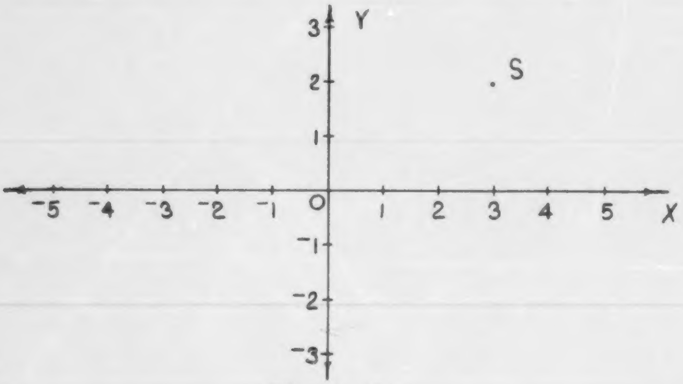

Y
3
2
1
S
⁻5
⁻4
⁻3
⁻2
⁻1
0
1
2
3
4
5
X
⁻1
⁻2
⁻3

Figure 3

Next draw a vertical line through S as in Figure 4.

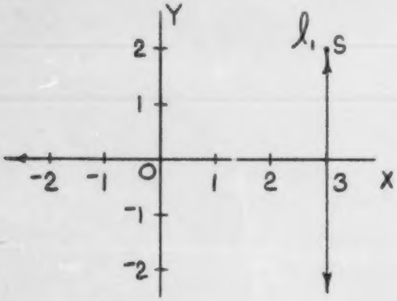


Figure 4

Look at the point where $\ell_1$ crosses the X-axis. The coordinate of this point is called the X-coordinate of S. In this case the X-coordinate of S is 3. Now look at Figure 5.

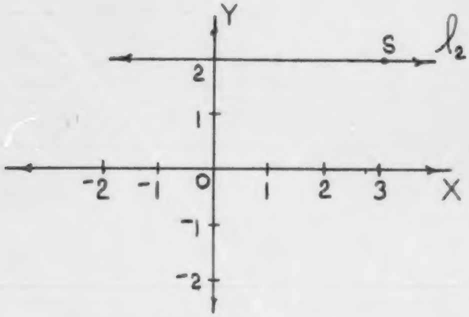


Figure 5

Here a horizontal line $\ell_2$ is drawn through S. The coordinate of the point where this line crosses the Y-axis is called the Y-coordinate of S. In Figure 6 you see the picture with both lines drawn just as it will appear on your paper when you do this kind of problem.

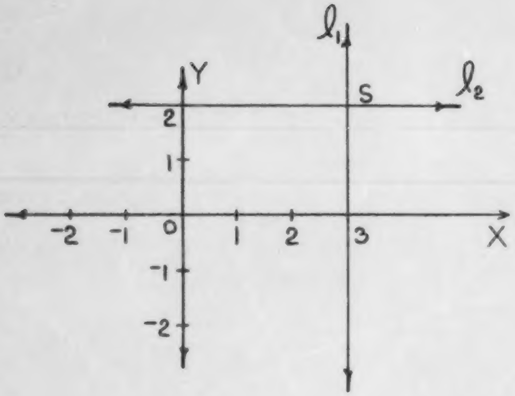

$l_1$
Y
S
$l_2$
2
1
-2
-1
0
1
2
3
X
-1
-2

Figure 6

### Exercises 19-2a
(Class Discussion)

Using the picture below fill in the table.

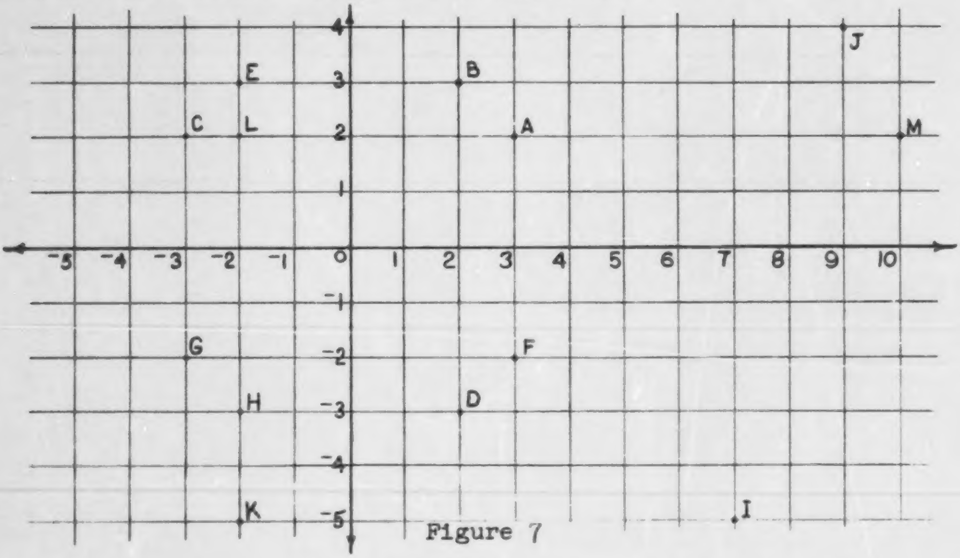

J
E
B
C
L
A
M
4
3
2
1
0
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9
10
-1
G
-2
F
H
-3
D
-4
K
-5
I

Figure 7

| Point | A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X-coordinate | | | | | | | | | | | | | |
| Y-coordinate | | | | | | | | | | | | | |

Naturally, we should like to find a shorter way of giving the position of a point. It takes too long to write:

> The X-coordinate of A is 3 and the Y-coordinate of A is 2.

Instead we shall simply denote A by (3, 2).

You might wonder how people can tell which number is the X-coordinate and which is the Y-coordinate.

It happens that people all over the world have agreed to put the X-coordinate first and the Y-coordinate second.

Drawing a point in the plane when its coordinates are given is called plotting the point. You will be asked to plot several points in the following exercises.

## Exercises 19-2b
(Class Discussion)

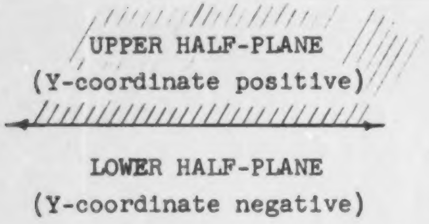
1. Copy the table and fill in the coordinates of the points shown in Figure 7.

| Point | A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Coordinates | (3, 2) | (2, 3) | | | | | | | | | | | |

2. a. What do the points E, L, H and K in Figure 7 have in common?
   b. What do their coordinates have in common?
3. a. What do the points C, L, A and M have in common?
   b. What do their coordinates have in common?
4. Plot the points whose coordinates are given below.

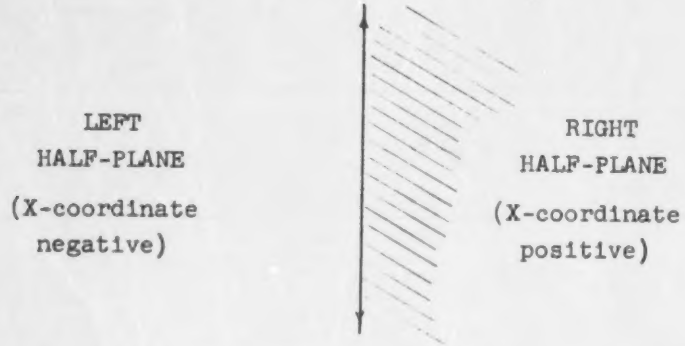
| Point | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Coordinates | (1, 4) | (6, 1) | (4, 4) | $(^-1, 3)$ | $(^-4, 3)$ | $(7, ^-3)$ | $(5, ^-4)$ | $(^-3, 4)$ | $(^-3, ^-4)$ | $(^-5, ^-1)$ | $(^-2, 3)$ |

You will remember that a line separates a plane into two half-planes. The X-axis separates the plane into two half-planes called the upper half-plane and the lower half-plane.


UPPER HALF-PLANE
(Y-coordinate positive)
LOWER HALF-PLANE
(Y-coordinate negative)

Do you see that:

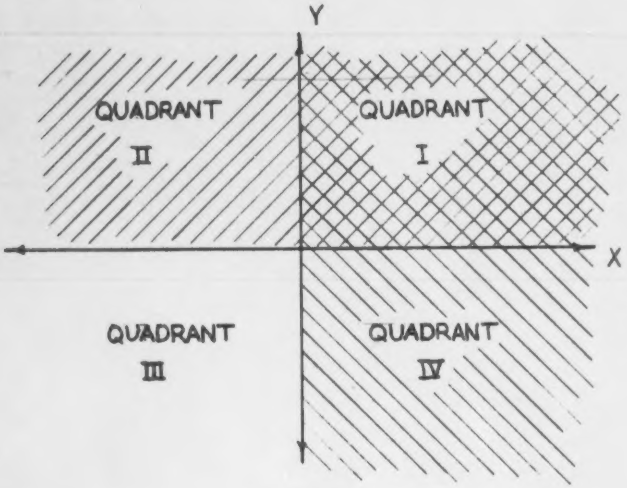
points in the upper half-plane have positive Y-coordinates?
points in the lower half-plane have negative Y-coordinates?
points on the X-axis have Y-coordinates equal to zero?


LEFT
HALF-PLANE
(X-coordinate
negative)
RIGHT
HALF-PLANE
(X-coordinate
positive)

You should see that:

points in the right half-plane have X-coordinate positive;
points in the left half-plane have X-coordinate negative;
points on the Y-axis have X-coordinate equal to zero.

If you place one of the last two pictures on top of the other, you can see that these four half-planes intersect in four regions. These regions are called quadrants.



These quadrants are simply the interiors of the four (right) angles formed by the two axes.

## Exercises 19-2c
(Class Discussion)

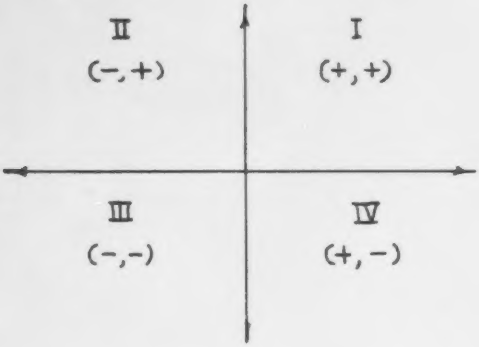
1. Points in the first quadrant have ________ X-coordinates since they lie in the ________ half-plane; and they have ________ Y-coordinates since they lie in the ________ half-plane.

2. Points in the second quadrant have ________ X-coordinates since they lie in the ________ half-plane; and they have ________ Y-coordinates since they lie in the ________ half-plane.

3. Points in the third quadrant have ________ X-coordinates since they lie in the ________ half-plane; and they have ________ Y-coordinates since they lie in the ________ half-plane.

4. Points in the fourth quadrant have ________ X-coordinates since they lie in the ________ half-plane; and they have ________ Y-coordinates since they lie in the ________ half-plane.

You may use a picture to help you remember these facts:



## Exercises 19-2d

1. Plot the following points and draw line segments connecting each point to the following one. Then connect the first and last points.

   A (0, 9), B (5, 5), C (2, 6), D (6, 2), E (2, 3),
   F (7, $^-2$), G (1, $^-2$), H (1, $^-7$), I ($^-1$, $^-7$), J ($^-1$, $^-2$),
   K ($^-7$, $^-2$), L ($^-2$, 3), M ($^-6$, 2), N ($^-2$, 6), O ($^-5$, 5).

2. Plot the following points and draw line segments connecting each point to the following one. Connect the first and last points.

   A (6, 1), B (8, 4), C (7, 0), D (10, $^-3$), E (6, $^-1$),
   F ($^-3$, $^-5$), G ($^-9$, $^-4$), H ($^-6$, $^-2$), I ($^-9$, $^-2$), J ($^-7$, 1),
   K (0, 4), L ($^-1$, 2).

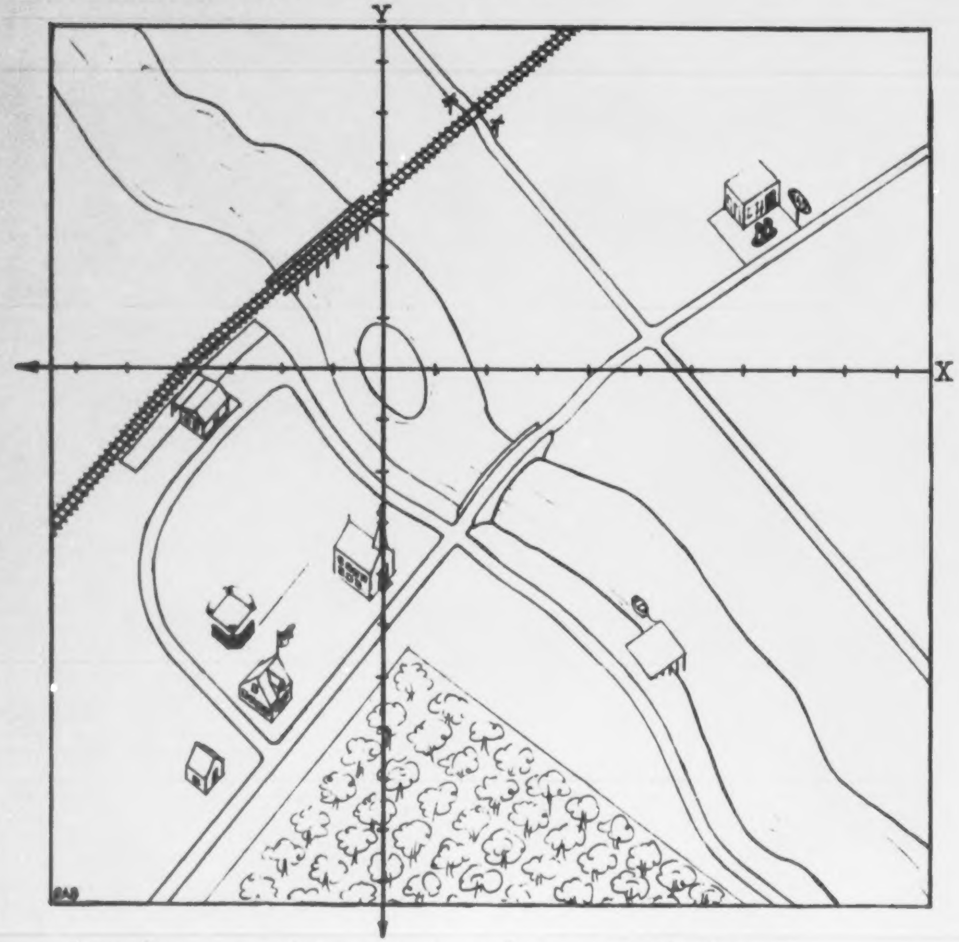
3. a. Plot the following points: A $(7, {}^-3)$, B $({}^-2, 10)$, C $({}^-2, {}^-6)$, D $(7, 7)$, E $({}^-8, 2)$.
   b. Draw the line segments: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$.

4. Without plotting, give the quadrants of the points whose coordinates are given below.
   a. $(3, 5)$
   b. $(1, {}^-4)$
   c. $({}^-4, {}^-4)$
   d. $({}^-3, {}^-1)$
   e. $(8, 6)$
   f. $({}^-7, 1)$
   g. $({}^-3, {}^-5)$
   h. $(100, 2)$
   i. $(31, {}^-31)$
   j. $({}^-7, {}^-72)$
   k. $({}^-51, 25)$
   l. $({}^-27, {}^-100)$

5. Plot the following points using the same coordinate plane:
   a. A $(\frac{7}{4}, 2)$
   b. B $({}^-\frac{1}{2}, {}^-\frac{1}{2})$
   c. C $(\frac{10}{3}, {}^-\frac{4}{3})$
   d. D $({}^-\frac{5}{2}, 0)$
   e. E $({}^-6, \frac{11}{2})$
   f. F $({}^-\frac{9}{4}, {}^-\frac{9}{2})$
   g. G $(0, \frac{13}{4})$
   h. H $(\frac{1}{2}, {}^-2)$
   i. I $({}^-6, {}^-\frac{3}{2})$
   j. J $(0, 0)$

6. Plot the following points using the same coordinate plane:
   a. A $(1.4, 5.3)$
   b. B $({}^-5.5, {}^-3.5)$
   c. C $(6, {}^-2.7)$
   d. D $({}^-3, 3.3)$
   e. E $(0, {}^-1.5)$

7. Here is a map with X- and Y-axes drawn on it and the "integer points" marked on the axes:



Give (as accurately as you can) the coordinates of:

a. the door of the church

b. the gasoline pumps

c. the ends of the highway bridge

d. the middle of the railroad bridge

e. the corner of the orchard

f. home-plate on the baseball diamond

g. the crossroads on the left bank of the river

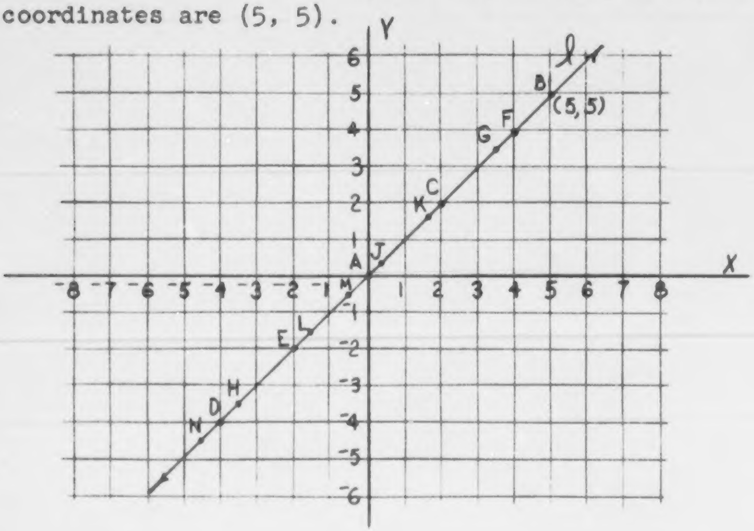
8. On the map in Problem 7 tell what you find at the following points:
   a. The origin
   b. (2, 5)
   c. (5.4, 0.7)
   d. $(5.5, {}^{-}5.5)$
   e. $({}^{-}3.5, {}^{-}1)$
   f. $({}^{-}2, {}^{-}6.5)$

## 19-3. Graphs in the Plane.

The study of geometry with the use of coordinates is called coordinate geometry. This branch of mathematics was started by the mathematician René Descartes in 1637. This invention was a great step forward in mathematics and made possible the discovery of calculus which followed shortly after. In the rest of this chapter you will learn about some very simple coordinate geometry.

### Exercises 19-3a
(Class Discussion)

On the coordinate plane below we have drawn the line $\ell$ through the point A (0, 0) and the point B (5, 5). The notation B (5, 5) means that B is a letter used to name the point whose coordinates are (5, 5).



1. Find the coordinates of some other points on this line.
   Copy and complete the table below:

| Point | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| X-coordinate | 0 | 5 | 2 | | | |
| Y-coordinate | 0 | 5 | | | | |

2. What do you notice about the coordinates of these points?
3. Now try some points whose coordinates are not integers.
   Estimate the coordinates as closely as you can.

| Point | G | H | J | K | L | M | N |
|---|---|---|---|---|---|---|---|
| X-coordinate | $3\frac{1}{2}$ | | | | | | |
| Y-coordinate | $3\frac{1}{2}$ | | | | | | |

4. Does the same relation between the X- and Y-coordinates that you noticed before still seem to hold?
5. What do you conclude about the coordinates of points on this line?

You should have noticed that for all the points that you checked in the last example the X- and Y-coordinates are equal. This relation holds true for all points on this line. Do you suppose that all points with X- and Y-coordinates equal will lie on this line? Try to find out. Plot these points on the above graph.

$P(1, 1)$, $Q(^-3, ^-3)$, $R(4.3, 4.3)$, $S(^-2\frac{2}{3}, ^-2\frac{2}{3})$.

Do these points seem to lie on the line $\ell$? It is true that all points whose X- and Y-coordinates are equal lie on the line $\ell$.

Here is another way of looking at what you have just developed. Let $x$ and $y$ be numbers. Then $(x, y)$ is a point somewhere in the plane. If $y = x$ then this point lies on the line $\ell$. If $y \neq x$ then this point does not lie on $\ell$.

This fact is expressed by saying that the line $\ell$ is the graph of all the points (x, y) for which $y = x$. (Sometimes people shorten this by saying: the line $\ell$ is the graph of the equation $y = x$.)

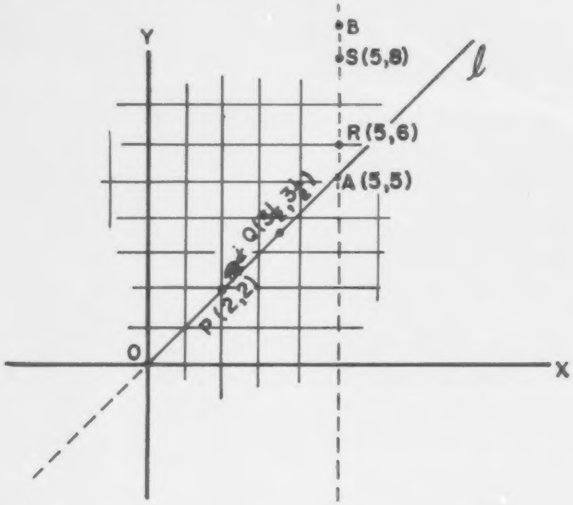
You have seen how the equation

$$y = x$$

identifies a certain line, $\ell$ . The line may be regarded as the "picture" of the equation. You can learn still more from this picture.

Look at the two half-planes determined by the line $\ell$ in the last example. There is more to be learned about the coordinates of points lying in these half-planes.

Consider a bug located at the origin, O, who wants to get to the point B lying above $\ell$ . This bug is allowed to move only along the line $\ell$ and along vertical lines. Let us trace the path that the bug takes.



The bug moves from the origin (0, 0) through point P(2, 2), then through point Q $(3\frac{1}{2}, 3\frac{1}{2})$ , and on to point A(5, 5). At point A, the bug starts to move along line $\overleftrightarrow{AB}$. He moves from A(5, 5) through R(5, 6), then through S(5, 8) and finally reaches point B.

The path of the bug consists of two segments, $\overline{OA}$ and $\overline{AB}$. Along $\overline{OA}$ the bug passed through point P(2, 2), point Q $(3\frac{1}{2}, 3\frac{1}{2})$, and point A(5, 5). Along $\overline{AB}$ the bug passed through point R(5, 6) and point S (5, 8).

Do you see that:

1. While the bug was on $\overline{OA}$ his Y-coordinate was equal to his X-coordinate?

2. When the bug moved upward (along $\overline{AB}$) his X-coordinate remained 5 but his Y-coordinate became greater than 5 and continued to increase all the way to B?

Now do you see that the Y-coordinate of B must be greater than the X-coordinate of B? You should see that for every point (x, y) lying in the half-plane above the line $\ell$ you have:

$$y > x$$

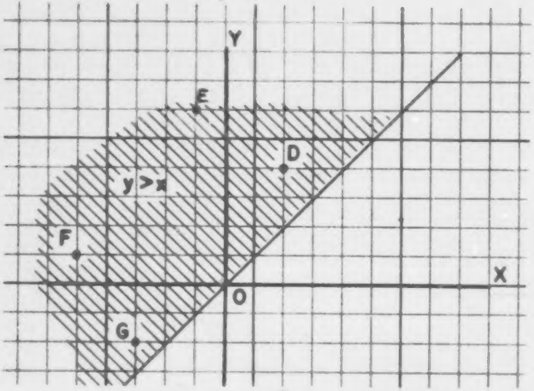


Figure 8

Test this conclusion by reading the coordinates of points D, E, F, and G. Do you find that the Y-coordinate of each of these points is greater than its X-coordinate?

You should also see that for every point (x, y) lying below the line $\ell$ you have

$$y < x$$

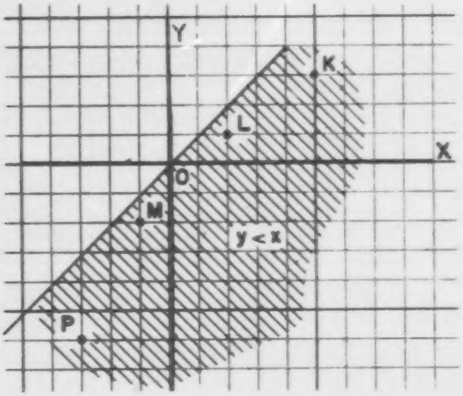


Figure 9

Test this conclusion by reading the coordinates of points K, L, M, and P. Do you find that the Y-coordinate of each of these points is less than its X-coordinate?

The shaded region in Figure 8 is the graph of the set of those points (x, y) for which $y > x$. The shaded region in Figure 9 is the graph of the set of those points (x, y) for which $y < x$.

## Exercises 19-3b

1. Consider the coordinates of the following points and tell whether the points lie on, above, or below the line $y = x$.

| | |
|---|---|
| a. A (4, 7) | f. F (1, $^-1$) |
| b. B (1, 1) | g. G ($^-3$, $^-4$) |
| c. C ($^-1$, 1) | h. H ($^-4$, $^-2$) |
| d. D (3, 1) | i. I ($^-7$, $^-7$) |
| e. E ($^-2$, 2) | j. J ($^-1$, $^-3$) |

2. a. Plot the points given in Problem 1.
   b. On the same coordinate plane draw the line $y = x$.
   c. Check your answers to Problem 1.
   d. Use a protractor and measure the angle formed by the part of the line in the first quadrant and the positive X-axis.

3. a. Plot the points $A(4, 4)$ and $B(2, 0)$ and draw $\overleftrightarrow{AB}$.
   b. On the same graph plot the points $C(6, 3)$ and $D(0, 1)$ and draw $\overleftrightarrow{CD}$.
   c. Estimate the coordinates of the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

4. Draw the line through $E(2, 1)$ and $F(2, 3)$ and describe its position in the plane.

5. Draw the line through $G(2, 3)$ and $H(5, 3)$ and describe its position in the plane.

6. Draw the line through $J(0, 0)$ and $K(2, 3)$.

---

## 19-4. Other Equations, Other Lines.

In the last section we took a certain line, $\ell$, and found that this line was the set of those points $(x, y)$ for which $y = x$. In this section we shall look at the same kind of problem the other way around. We shall take the equation

$$y = 2x$$

and try to find the "picture" or graph of those points $(x, y)$ in the coordinate plane whose coordinates satisfy this condition.

We shall find a few coordinate pairs which satisfy the condition, plot them, and then make a guess as to what the graph should look like.

## Exercises 19-4a

(Class Discussion)

1. For each of the points in the table the X-coordinate is given. Find the Y-coordinate for each point so that the Y-coordinate is twice the X-coordinate.

| Point | A | B | C | D | E | F | G | H | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X-coordinate | 3 | $^{-}4$ | 2 | $^{-}3$ | 0 | 4 | $2\frac{1}{2}$ | $^{-}3\frac{1}{2}$ | $^{-}1\frac{2}{3}$ | 1 | $^{-}2$ |
| Y-coordinate | 6 | $^{-}8$ |  |  |  |  |  |  |  |  |  |

2. Plot the points in Problem 1 on a piece of graph paper.

3. On what kind of curve do all these points seem to lie? Draw this curve.

4. Check to see whether some other points on the graph drawn in Problem 3 satisfy the condition $y = 2x$. How about the point with X-coordinate $^{-}1$? What is the Y-coordinate of this point? Do the coordinates of this point satisfy $y = 2x$? Select some points yourself and make this test.

5. What do you conclude about the set of those points $(x, y)$ for which $y = 2x$?

6. a. Use another sheet of graph paper and try to find the graph of those points $(x, y)$ for which $y = 3x$. Fill in the following table and plot the points.

| Point | A | B | C | D | E | F | G | H | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| X-coordinate | 2 | $^{-}3$ | 1.5 | $\frac{2}{3}$ | $^{-}2$ | $^{-}1$ | $^{-}2\frac{2}{3}$ | $2\frac{2}{3}$ | $^{-}0.5$ | 0 |
| Y-coordinate | 6 | $^{-}9$ | 4.5 |  |  |  |  |  |  |  |

   b. On what kind of curve do these points seem to lie? Draw this curve carefully.

   c. Select a few more points on this curve and find the coordinates of these points. Do the coordinates seem to satisfy the condition $y = 3x$?

7. Find the graph of $y = \frac{1}{2}x$ in the same way.

   a. Make a table as in Problem 6 using as X-coordinates:

   $0,\ 4,\ {}^{-}10,\ 7,\ {}^{-}2\frac{2}{3},\ {}^{-}5,\ 2.8,\ {}^{-}6.6,\ 1.2,\ {}^{-}(\frac{1}{2}).$

   Find the Y-coordinates using $y = \frac{1}{2}x$.

   b. Plot these points and draw the curve through them.

   c. Check some other points on this curve. Do their coordinates seem to satisfy $y = \frac{1}{2}x$?

8. Find the graph of the equation $y = \frac{2}{3}x$ :

   a. Check that the coordinates of the points (0, 0) and (9, 6) satisfy this equation. Plot these points and carefully draw the line through them.

   b. Select five other points on this line taking some points on each side of the origin. Do the coordinates of these points seem to satisfy the equation $y = \frac{2}{3}x$?

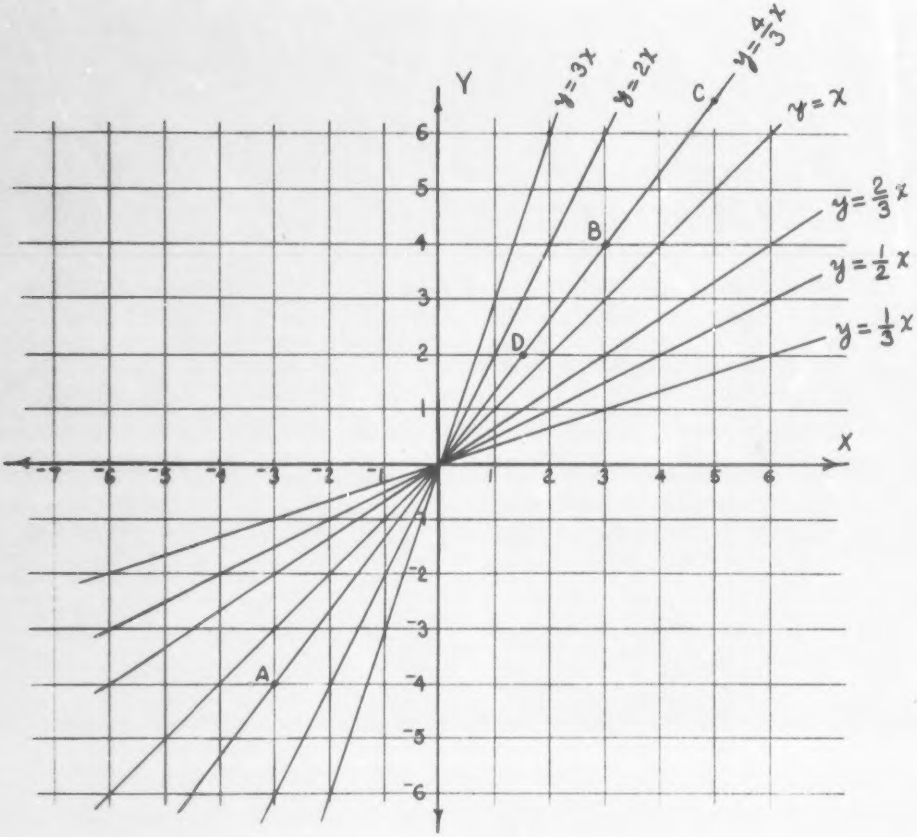
In the above exercises you found that the graphs of the equations

$$y = 2x, \quad y = 3x, \quad y = \frac{1}{2}x, \quad y = \frac{2}{3}x$$

were all straight lines passing through the origin. These results can be generalized in the following way:

for any number $b$,
the set of points $(x, y)$ for which $y = bx$
is a straight line through the origin.

All the graphs you have drawn in the last set of exercises, and some others as well, are shown in the following figure.



You should check a number of points on each of those lines to see whether the coordinates of these points really satisfy the equation with which the line is labeled. A few points have been marked for you to check on the line $y = \frac{4}{3}x$. What are the coordinates of A, B, C, and D? Do these coordinates satisfy $y = \frac{4}{3}x$ ?

Lines through the origin give us a convenient way of using graphs to do multiplication problems. This is useful when we have a whole list of numbers to be multiplied by the same factor. For example, suppose you are given a list of elevations of geographical points in kilometers which you wish to change to elevations in miles. To change elevations in kilometers to elevations in miles we must multiply the number of kilometers by $0.62$ ; the result is then the elevation in miles. In other words, if $x$ is the elevation of a certain point in kilometers and $y$ is the elevation of the same point in miles, we then have

$$y = (0.62)x.$$

This leads us to draw the graph of the equation $y = (0.62)x$. We do this as follows:

1. First, we take a convenient value of $x$, say $x = 10$.
2. Next, we compute from the equation the corresponding value of $y$. The result is $y = 6.2$.
3. Now we know that the point $(10, 6.2)$ lies on the graph of the equation $y = (0.62)x$. We therefore plot this point.
4. The origin $(0, 0)$ also lies on the line $y = 0.62x$. We also plot this point.
5. Since two points determine a line, we need to plot no further points in order to draw the graph of $y = (0.62)x$. We draw the line through $(0, 0)$ and $(10, 6.2)$.

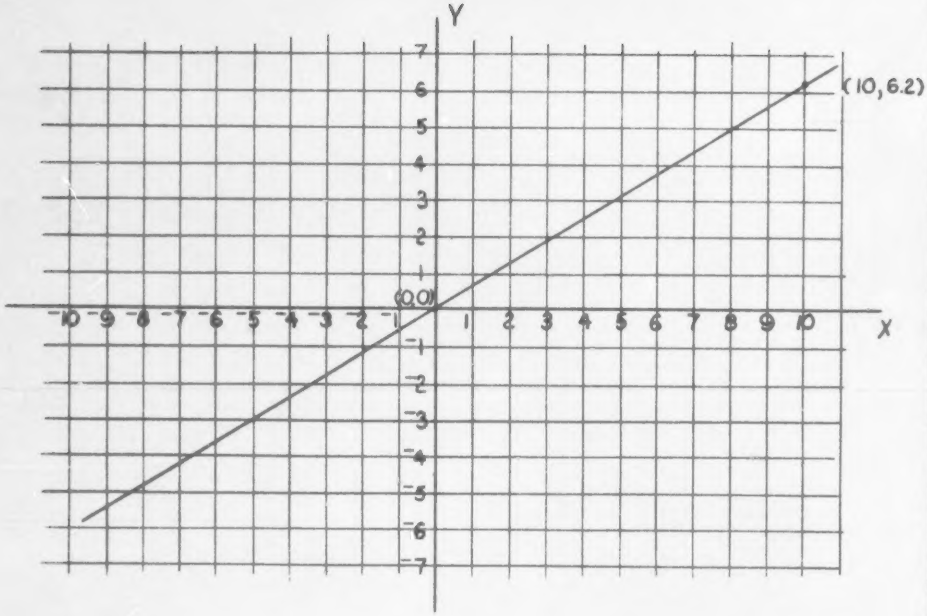


Figure 10

Now we can use this graph to find elevations in miles when elevations in kilometers are given. Here is a table of elevations of geographical locations with elevations in kilometers given. Fill in the blank spaces.

| GEOGRAPHICAL LOCATION | ELEVATION IN KILOMETERS | ELEVATION IN MILES (as read from Figure 10) | ELEVATION IN MILES obtained by multiplying elevation in kilometers by 0.62 |
|---|---|---|---|
| Matterhorn, Switzerland | 4.5 | | |
| Mount Everest | 8.9 | | |
| Dead Sea | ⁻0.4* | | |
| Deepest Point in Pacific | ⁻11.0 | | |
| Mount Whitney | 6.2 | | |
| Deepest Oil Well | ⁻5.0 | | |
| Deepest Ocean Descent by Man | ⁻10.9 | | |

*Negative Elevations denote points below sea level.

You should not try to read the coordinates from the graph any closer than the nearest tenth. This has the effect of "rounding" the answer. This explains why you may not get exactly the same result which you get by multiplying.

Until now we have been doing multiplication by a number b graphically only when the number b is positive. The method works just as well when b is negative.

### Exercises 19-4b
(Class Discussion)

Let us draw the graph of

$$y = (^{-}2)x.$$

1. First find the Y-coordinates of the points on this graph for which the X-coordinates are given.

| Point | X-coordinate | Y-coordinate |
|---|---|---|
| A | 0 | |
| B | 4 | |

2. Plot the points A and B on the coordinate plane.
3. Draw the line through A and B. This is the graph of the set of those points $(x, y)$ for which $y = (^{-}2) \cdot x$.
4. Use your graph to find the following products.

| $x$ | 1 | 3 | 2.5 | $^{-}3$ | $^{-}1.5$ | 3 | $^{-}2$ | $^{-}2.5$ | $^{-}4$ |
|---|---|---|---|---|---|---|---|---|---|
| $(^{-}2) \cdot x$ | | | | | | | | | |

## Exercises 19-4c

1. a. Draw the graph of $y = 2x - 1$ by following these steps: Find the Y-coordinates of points for which the X-coordinates are given below.

   **Example:** When $x = {}^{-}4$, $y = 2({}^{-}4) - 1 = {}^{-}9$.

| X-coordinate | $^{-}4$ | $^{-}3$ | $^{-}2$ | $^{-}1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| Y-coordinate | $^{-}9$ | | | | | | | | |

   b. Plot the points whose coordinates you found in Part (a).

   c. On what kind of curve do these points lie?

2. Answer the questions in Problem 1 for the same X-coordinates and for the following equations:

   a. $y = x + 4$

   b. $y = {}^{-}2x + 1$

   c. $y = {}^{-}2x - 1$

3. Given the equation $y = \frac{1}{4}x^2$ :

   a. Find the Y-coordinates for the points for which the X-coordinates are given in the table below.

| X-coordinate | $^{-}4$ | $^{-}3$ | $^{-}2$ | $^{-}1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| Y-coordinate | | | | | | | | | |

   b. Plot the points whose coordinates you found in Part (a).

   c. Do these points seem to lie on a curve? Sketch this curve carefully.

4. Consider the equation $y = \frac{12}{x}$ .

   a. Find the Y-coordinate for the points for which the X-coordinates are given in the table below.

| X-coordinate | $^{-}12$ | $^{-}9$ | $^{-}6$ | $^{-}4$ | $^{-}3$ | $^{-}2$ | $^{-}1$ | 0 | 1 | 2 | 3 | 4 | 6 | 9 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Y-coordinate | | | | | | | | | | | | | | | |

   b. Plot the points whose coordinates you found in Part (a).

   c. On what kind of curve do these points lie? Try to sketch this curve.

5. Bill and Don put a pan of ice on the stove. The pan held both Fahrenheit and Centigrade thermometers. As the pan heated Bill read the Fahrenheit thermometer several times. Don read the Centigrade thermometer each time also. Their readings are given in the table below.

| Don | 0°C | 20°C | 45°C | 70°C | 100°C |
|---|---|---|---|---|---|
| Bill | 32°F | 68°F | 113°F | 158°F | 212°F |

   a. On graph paper draw coordinate axes so that just the first quadrant appears. Choose a scale on the axes for numbers from 0 to 225.
   b. Let the numbers on the X-axis represent degrees Centigrade and the numbers on the Y-axis represent degrees Fahrenheit. Plot each point whose Y-coordinate is Don's reading and whose X-coordinate is Bill's corresponding reading.
   c. On what sort of curve do these points seem to lie? Draw this curve carefully.
   d. From this curve can you read what Centigrade temperature corresponds to 98.6°F (normal body temperature)?

6. In Problem 5, you found that the relationship between Centigrade and Fahrenheit temperature can be graphed as a straight line.

   Use the points (0, 32) and (100, 212) and draw the line through them. Continue the line into the other quadrants.
   a. Does the line pass through every quadrant?
   b. What is the Centigrade temperature that corresponds to 0° F?
   c. The hottest weather recorded was 136° F in Azizia, Tripolitania, in 1922. What is the corresponding Centigrade temperature?
   d. The coldest weather recorded was $^{-}90^{\circ}$ F in Siberia in 1892. What would a Centigrade thermometer have read?

7. Insurance companies are interested in letting people know the stopping distance of a car traveling at different speeds.

| Speed (mph.) | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| Stopping distance (ft.) | 16 | 42 | 78 | 124 | 180 | 246 |

a. Plot these points on a graph.

b. Draw a smooth curve through all these points. Is the curve a straight line?

c. According to your graph what distance is needed for a car to stop if:

its speed is 25 mph?
its speed is 37 mph?
its speed is 45 mph?

d. Speeding automobiles often cause accidents. After such an accident a policeman or insurance company representative may try to determine how fast a car was traveling by measuring the length of the skid marks. How fast may he assume that the car was traveling if the skid marks were measured to be 200 ft? 60 ft?

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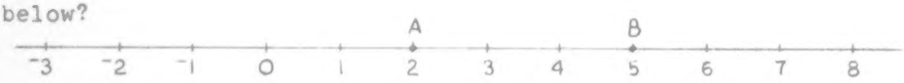
## 19-5. Distance.

The distance between two points on a line, in the plane, or in space is the length of the line segment between them. In expressing the length of such a segment, we state a number (the measure of length) and the unit of length, as in 5 inches or 7 feet. But we shall consider only the case in which the two points are on a number line or a coordinate plane. Therefore, the unit will not be mentioned when the distances are given.

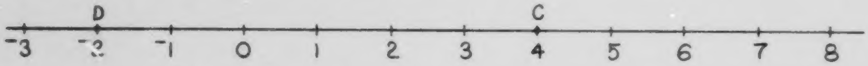
What is the distance between the points A and B shown below?



We can find the result by counting the number of units from A to B. We find the answer to be 3. We could also have obtained this result by subtracting the coordinate of A from that of B:

$$5 - 2 = 3.$$

What is the distance between C and D below?



Again we can count the units between C and D and find that the distance is 6. Or we could subtract the coordinate of D from that of C,

$$4 - (^-2),$$

and again obtain 6.

We see that the distance between two points on the number line is the difference of the coordinates of these points--the greater minus the smaller.

The same method can be applied to finding distances between pairs of points in the coordinate plane in certain special cases.

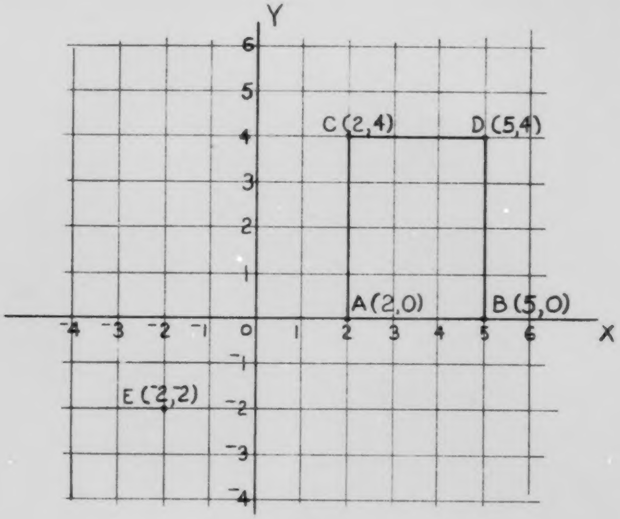


Figure 11

Let us find the distance between A(2, 0) and B(5, 0) in Figure 11. The X-coordinates of these points are just the coordinates of these same points on the X-axis when we consider the X-axis as a number line. So the distance between A(2, 0) and B(5, 0) is 5 - 2, the difference of the X-coordinates of these points.

Now let us find the distance between the points C(2, 4) and D(5, 4). You can see that the points ABDC form a rectangle. The opposite sides $\overline{AB}$ and $\overline{CD}$ therefore have equal lengths. We know that the length of $\overline{AB}$ is the difference of the X-coordinates of A and B. But the X-coordinates of C and D are respectively the same as those of A and B. Therefore, the distance between C and D is the difference of the X-coordinates of C and D, or 5 - 2. You should be able to see that:

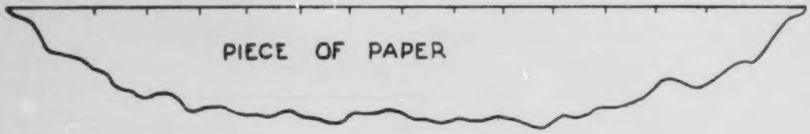
> If two points lie on the same horizontal line (or what amounts to the same thing, if they have the same Y-coordinate), then the distance between them is the difference of their X-coordinates (the greater minus the smaller.)

Similarly:

> If two points lie on the same vertical line (or what amounts to the same thing, if they have the same X-coordinate), then the distance between them is the difference of their Y-coordinates.

Now we know how to find the distance between two points if they lie on the same horizontal line or if they lie on the same vertical line. But how can we handle other situations? For example, how can we find the distance between the points A and D in Figure 11? or between the points C and E?

Here is one thing we can do. We can make a ruler by marking off on the edge of a sheet of paper the units used in constructing our coordinate system.



Then we can use this ruler to measure distances between points in Figure 11. We show how to measure the distance between C and E in the following figure:

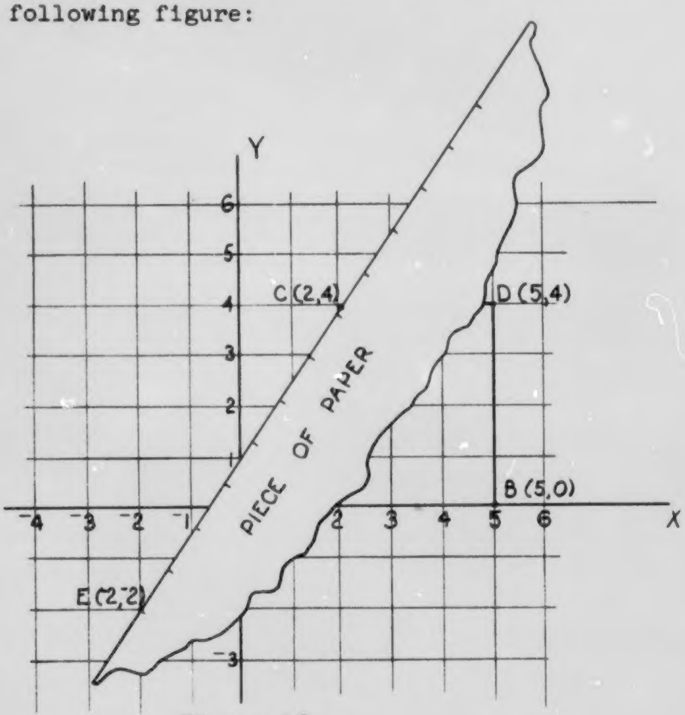


Figure 12

We see that the distance between C and E is approximately 7.2. Measure the distance between A and D in Figure 12. In some ways this method of finding distances is not very good. When we were given the coordinates of two points on the same vertical or horizontal line we could find the exact numerical value of the distance. When we have to measure to find distances, the accuracy of our answer depends on the accuracy with which our points are plotted and the accuracy with which we can read our ruler. After we have read the next two sections, we shall be ready to work out a method for finding the exact numerical value of the distance between any two points.

## Exercises 19-5

1. Find the distance between the following pairs of points without plotting. Tell whether the points lie on a vertical line or on a horizontal line.
    a. (0, 0) and (5, 0)
    b. (2, 0) and (2, 7)
    c. (3, 4) and (8, 4)
    d. (3, 4) and (3, 9)
    e. (2, 7) and (2, $^-1$)
    f. ($^-6$, $^-1$) and ($^-3$, $^-1$)
    g. ($^-8$, 2) and (7, 2)
    h. ($^-4$, $^-8$) and ($^-4$, $^-6$)
2. Plot the pairs of points given in Problem 1 and check your answers to Problem 1 on your graph.
3. a. Draw the triangle whose vertices are the points A(4, 0), B(0, 3), C(0, 0).
    b. What is the length of $\overline{AC}$?
    c. What is the length of $\overline{BC}$?
    d. Make a ruler like the one described in this section and measure the length of $\overline{AB}$.
4. a. Plot the points A(0, 0), B(6, 0), and C(3, 4).
    b. Draw $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ and measure these distances with a ruler made out of coordinate paper.
5. a. Plot the points A(2, 2), B($^-2$, 2), C($^-2$, $^-3$), and D(2, $^-3$).
    b. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, and find their lengths.
    c. What kind of curve is this figure?
    d. Draw the diagonals.
    e. What are the coordinates of the point of intersection of the diagonals?

6. a. Plot the points $S(2, 1)$, $T(3, 3)$, $U(^-2, 3)$, and $V(^-3, 1)$.
   b. Draw $\overline{ST}$, $\overline{TU}$, $\overline{UV}$, and $\overline{VS}$.
   c. What is the distance from $S$ to $V$?
   d. What is the distance from $T$ to $U$?
   e. What is the name of the quadrilateral $STUV$?
   f. Draw the diagonals and find the coordinates of their point of intersection from the graph.

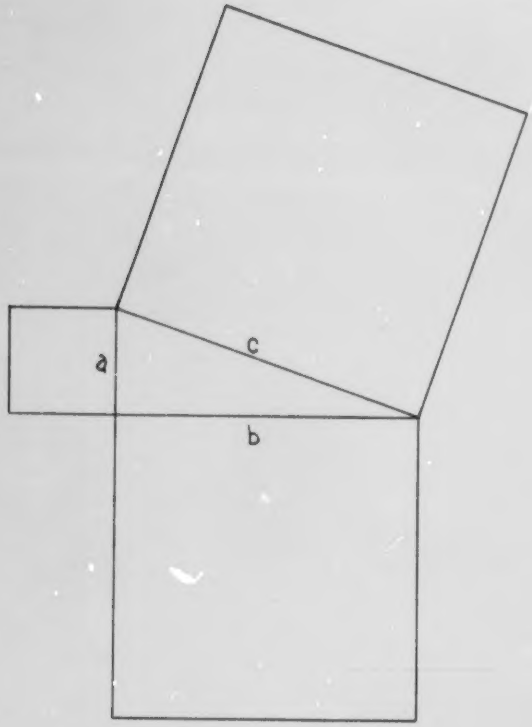
7. The vertices of a trapezoid are at the points $(^-4, 3)$, $(0, 0)$, $(3, 0)$ and $(7, 3)$. Plot these points. What are the lengths of the parallel sides of the trapezoid?

---

## 19-6. A Property of Right Triangles.

In this section, and in the next, we are going to study a famous property of right triangles. It is important in elementary mathematics as well as in the most advanced mathematics. It is also very interesting and simple to state. Take a right triangle

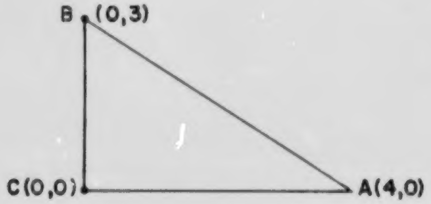
and place a square on each side of its edges as shown:



Then the area of the square placed on the hypotenuse (the side opposite the right angle) is the sum of the areas of the other squares. If the lengths of the sides are a, b, and c, with c being the length of the hypotenuse, then the statement becomes

$$a^2 + b^2 = c^2.$$

You can check the truth of this statement in some simple cases. First locate the point A(4, 0) and the point B(0,3) on the coordinate plane.



Let the point C be the origin (0, 0). Now the angle ACB is a right angle so that the triangle ACB is a right triangle with its segment $\overline{AB}$ as hypotenuse. The sides $\overline{AC}$ and $\overline{BC}$ have lengths 4 and 3. If you now measure the length of $\overline{AB}$ you will find that this length seems to be exactly 5. It is easy to check that

$$3^2 + 4^2 = 5^2.$$

This shows that the relation

$$a^2 + b^2 = c^2$$

seems to hold in this case, but you have not proved it. If the true length of $\overline{AB}$ were 5.01 you would not have detected it in your measurement. In that case

$$a^2 + b^2 \quad \text{and} \quad c^2$$

would have been nearly equal but not quite. The accuracy to which $a^2 + b^2$ and $c^2$ agree in your drawing depends on the accuracy of your drawing and your measurement.

The first person known to have given a proof of this property was the Greek mathematician Pythagoras, who lived about 500 B.C. (nearly 2500 years ago!). Because he was the first to prove it, the property is still known by his name; it is called the Pythagorean Property.

Pythagoras proved that: in any right triangle, the area of the square on the hypotenuse (longest side) is equal to the sum of the areas of the squares on the other two sides.

In the next section, you will see a proof of the Pythagorean Property, but it will not be the proof given by Pythagoras.

## Exercises 19-6

1. In each of the following cases, two vertices of a right triangle are given. The third is C(0, 0). Draw each triangle. Measure $\overline{AB}$ as carefully as you can and check to determine whether $a^2 + b^2 = c^2$.

| | | | | | |
|---|---|---|---|---|---|
| a. | A(12, 0) | B(0, 5) | c. | A(24, 0) | B(0, 7) |
| b. | A(15, 0) | B(0, 8) | d. | A(21, 0) | B(0, 20) |

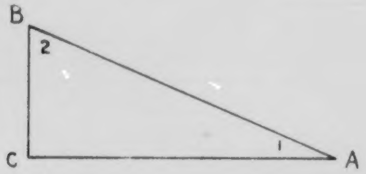
2. In the following problems $\overline{AB}$ is not a whole number. Measure $\overline{AB}$ to the nearest tenth and see how closely $a^2 + b^2$ and $c^2$ agree.
    a. A(3, 0) B(0, 2)
    b. A(5, 0) B(0, 4)
    c. A(7, 0) B(0, 5)

---

## 19-7. A Proof of the Pythagorean Property.

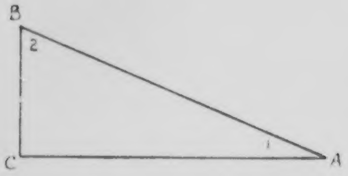
The purpose of this section is to prove the Pythagorean Property. Before we can begin this proof two simple facts must be pointed out.

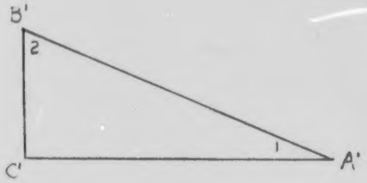
First let us consider a right triangle ABC



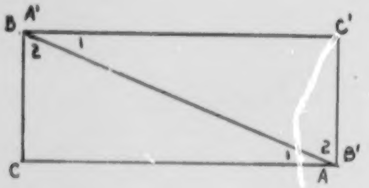
with the right angle at C. The sum of the measures of the three angles of the triangle is 180 and the measure of the angle C is 90.

Therefore the sum of the measures of angles A and B (the angles marked 1 and 2) is 90. Now cut out an exact copy of Δ ABC, call it Δ A'B'C'.





The two angles marked 1 will have equal measures. So will the two angles marked 2. Now move the triangle A'B'C' until A' coincides with B, and B' coincides with A, and C and C' are on opposite sides of $\overleftrightarrow{AB}$, like this:

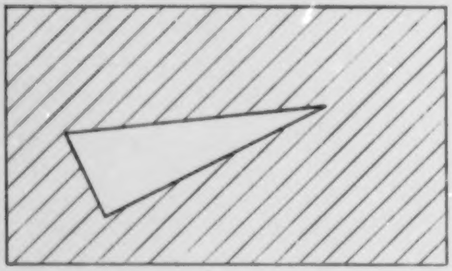


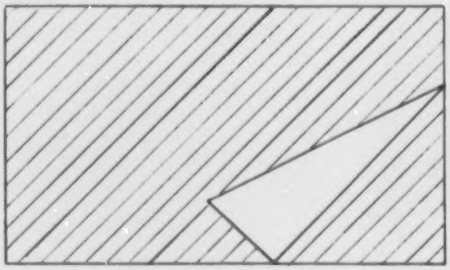
We see that the figure we have obtained is a rectangle because each of the corner angles is a right angle. This is true because we have already shown that an angle marked 1 and an angle marked 2 have the sum of their measures equal to 90.

That is the first of the simple facts which we need. The second is even simpler. It is this. Suppose you have a region, say a rectangular region as shown.

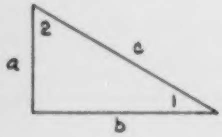
And suppose you make a cut-out of another region, say a triangular one, as shown.

And suppose you place this cut-out on top of the rectangular region. Then the area of the part of the rectangular region remaining uncovered will be the same, no matter where the triangle is placed.

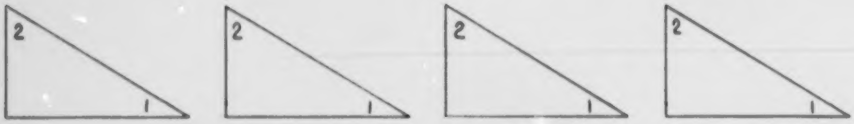




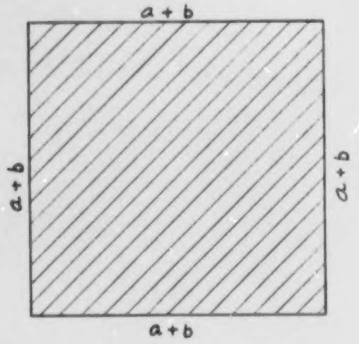
With this ground-work laid we are ready to prove the Pythagorean Theorem. First take any right triangle with sides of length a, b, and c, where c is the hypotenuse.



Cut out four exact copies of this triangle:



Next draw a square with side $a + b$.



Place the 4 cut-out triangles on the square in two different ways like this:

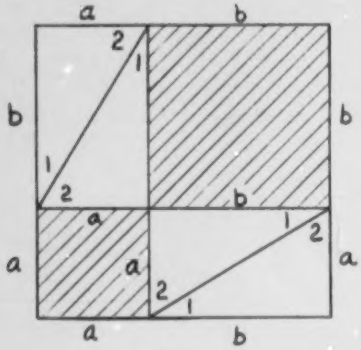


Figure 13

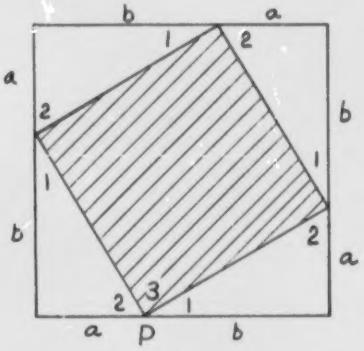


Figure 14

As we have observed, the shaded regions will have the same area in the two cases. In Figure 13 we see that the shaded area is the sum of the areas of a square of side $a$ and a square of side $b$. We shall show that the shaded region in Figure 14 is a square with side $c$.

Since this region has 4 sides each of length $c$ all we have to show is that each angle has a measure of 90. Look at Figure 14. Note the three angles marked 1, 2 and 3 having $P$ as vertex. We know that the sum of the measures of these angles is 180. But the sum of the measures of the angles marked 1 and 2 is 90. This leaves 90 for the measure of the angle marked 3. In the same way we see that the other angles of the shaded region in Figure 14 are right angles. This region is therefore a square with side-length $c$.

Now we have shown that

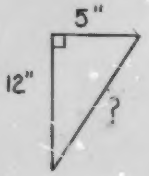
$$a^2 + b^2 = c^2.$$

## Exercises 19-7

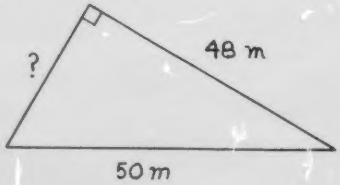
1. Find $c^2$ if $c^2 = a^2 + b^2$ and
   a. $a = 3$, $b = 4$
   b. $a = 5$, $b = 12$
   c. $a = 24$, $b = 7$
   d. $a = 10$, $b = 10$
2. Find $a^2$ if $c^2 = a^2 + b^2$ and
   a. $b = 9$, $c = 15$
   b. $b = 20$, $c = 25$
   c. $b = 20$, $c = 29$
   d. $b = 1$, $c = 15$
3. Find the number whose square is
   a. 64
   b. 225
   c. 169
   d. 144
   e. 100
   f. 900
   g. 10,000
   h. 1600
   i. 121
   j. 2500
4. A square piece of land contains 8100 square yards. What is the length of one side?
5. If two legs of a right triangle are 18 inches and 24 inches long, what is the length of the hypotenuse?
6. What is the length of the diagonal of a rectangle whose length is 8 feet and whose width is 6 feet?

7. The following figures are all right triangles. Find the length of the missing side.

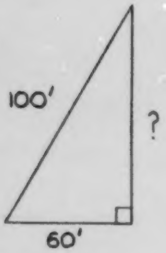
a.



b.



c.



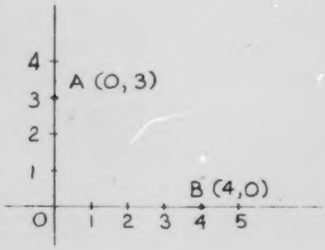
8. BRAINBUSTER. A packing box has a rectangular base 16 in. by 30 in. Can you pack a gun $34\frac{1}{2}$ in. long in the box so that it will lie flat on the bottom?

---

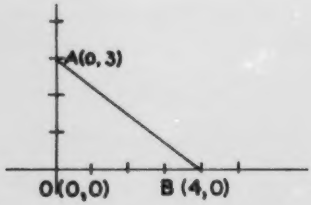
## 19-8. Back to Distance.

Now that we know the Pythagorean Property we are ready to return to the problem of distance in the plane. This time we shall solve the problem completely.

Let us try to find the distance between A(0, 3) and B(4, 0).



These two points, together with the origin, O, are the vertices of a right triangle.



The lengths of two of the sides $\overline{OA}$ and $\overline{OB}$ are seen to be 3 and 4. The length of the hypotenuse $\overline{AB}$ is the distance between A and B. Now we can use the Pythagorean Property. It tells us that

$$AB^2 = OA^2 + OB^2$$

or

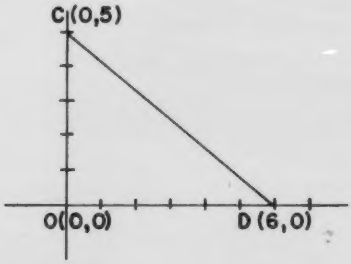
$$AB^2 = 3^2 + 4^2 = 9 + 16 = 25.$$

And now since the square of the number AB is 25 we see that

$$AB = 5.$$

Check this by measuring.

Let us try another problem. Let us find the distance between C(0, 5) and D(6, 0).



Again we have

$$CD^2 = OC^2 + OD^2$$

or

$$CD^2 = 5^2 + 6^2 = 25 + 36 = 61.$$

We see that $CD^2 = 61$ but what is CD itself? Do you know a number whose square is 61? We handle this by writing

$$CD = \sqrt{61}.$$

We read this as CD equals the square root of 61. The symbol "$\sqrt{61}$" is a name for the positive number whose square is 61. Similarly, the symbol "$\sqrt{5}$" is a name for the positive number whose square is 5, and "$\sqrt{4}$" is a name for the positive number whose square is 4.

But what is $\sqrt{61}$? It is easy to see that $\sqrt{4} = 2$ because $2^2 = 4$ and $\sqrt{9} = 3$ because $3^2 = 9$ and $\sqrt{25} = 5$ because $5^2 = 25$. But there is no whole number whose square is 61. We shall learn more about numbers like $\sqrt{61}$ in the next chapter. For use now, there is a table at the end of this chapter which gives decimal approximations to the square roots of counting numbers from 1 to 100. This table gives approximations to the nearest thousandth. For example, opposite 61 in the table of square roots we find 7.810. This tells us that

$$7.8095 < \sqrt{61} < 7.8105.$$

You could have found this without the table by noting that

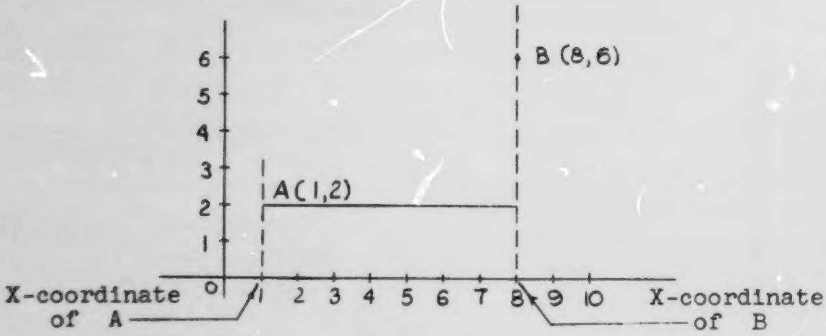
$$7^2 = 49 < 61 \qquad \text{and} \qquad 8^2 = 64 > 61$$

so that $$7 < \sqrt{61} < 8.$$

And $$(7.8)^2 = 60.84 < 61 \qquad \text{and} \qquad (7.9)^2 = 62.41 > 61$$

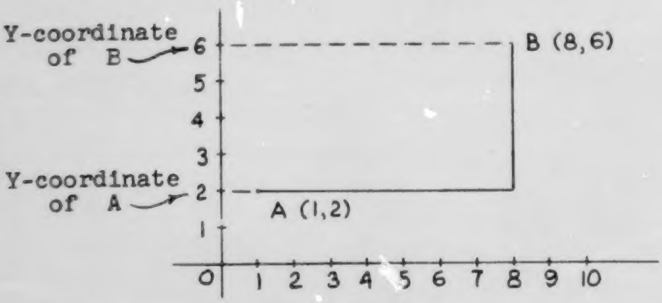
so that $$7.8 < \sqrt{61} < 7.9 \qquad \text{and so on.}$$

Now we are ready to consider the general case. We shall try to find the distance between the points $A(1, 2)$ and $B(8, 6)$. For this purpose let us suppose we have a bug which can move only on vertical and horizontal lines. Put the bug down at point A and let him move to point B. How can he do this?


B (8,6)
A(1,2)
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 9 10
X-coordinate of A
X-coordinate of B

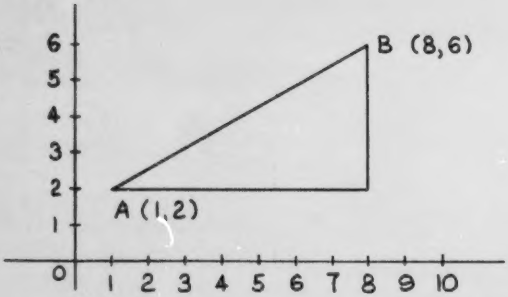
As you can see he will first travel horizontally until he reaches the vertical line through B. How far will he travel horizontally? Can you see that this distance is the difference of the X-coordinates of the points A and B, or 8 - 1?

Next he will turn the corner and proceed along this vertical line to B.


Y-coordinate of B
B (8,6)
6
5
4
3
Y-coordinate of A
2
A (1,2)
1
0
1 2 3 4 5 6 7 8 9 10

How far will he travel vertically? Do you see that this distance is the difference of the Y-coordinates of A and B, namely 6 - 2?

Finally we see that the path of the bug consists of a horizontal segment and a vertical segment. Moreover these two segments form the sides of a right triangle which has the segment $\overline{AB}$ as hypotenuse.



Since we know the lengths of the other two sides, we can use the Pythagorean Property to calculate the length of the hypotenuse. We have:

$$AB^2 = (8 - 1)^2 + (6 - 2)^2 = 7^2 + 4^2 = 49 + 16 = 65$$

and therefore

$$AB = \sqrt{65} .$$

Now, using the table of square roots at the end of the chapter, we find that $AB$ is approximately 8.062.

Collecting what we have learned from this example so that we can use it in other cases, we have:

1. Every segment $\overline{AB}$ is the hypotenuse of a right triangle having one horizontal side and one vertical side.
2. The length of the horizontal side is the difference of the X-coordinates of A and B. The length of the vertical side is the difference of the Y-coordinates of A and B.
3. The Pythagorean Property therefore gives us:
   $AB^2 = (\text{diff. of X-coords.})^2 + (\text{diff. of Y-coords.})^2$.

## Exercises 19-8

1. The points A, B and C are the vertices of right triangles. Find the lengths of the sides and draw the triangle in each case.
   - a. A(0, 6) B(8, 0) C(0, 0)
   - b. A(1, 2) B(4, 6) C(4, 2)
   - c. A(14, 3) B(2, 8) C(2, 3)
   - d. A(7, 0) B(22, 20) C(7, 20)
   - e. A(1, $^-2$) B(5, 1) C(5, $^-2$)

2. Find the distance between the points in each case, and plot the points on graph paper. If the distance is not an integer use the square root symbol to express your answer.
   - a. (40, 0) and (0, 30)
   - b. (20, 0) and (0, 21)
   - c. (2, 0) and (0, 3)
   - d. (2, 0) and ($^-7$, 0)
   - e. (1, 1) and (4, 5)
   - f. (8, 8) and (32, 1)
   - g. (4, 4) and ($^-4$, 10)
   - h. (1, 1) and (5, 7)

3. Give common names for each of the following:
   - a. $\sqrt{9}$
   - b. $\sqrt{25}$
   - c. $\sqrt{49}$
   - d. $\sqrt{169}$
   - e. $\sqrt{400}$
   - f. $\sqrt{1}$
   - g. $\sqrt{10,000}$
   - h. $\sqrt{3600}$
   - i. $\sqrt{324}$
   - j. $\sqrt{289}$

4. Fill in the blanks with whole numbers.
   - a. Since $2^2$ = ____ and $3^2$ = ____, if $c^2 = 6$ then c must be a number greater than ____ and less than ____.
   - b. Since $8^2$ = ____ and $9^2$ = ____, if $c^2 = 72$, then c must be a number greater than ____ and less than ____.

5. Locate each of the following between two consecutive integers: (For example: $4 < \sqrt{19} < 5$ since $4^2 = 16$ and $5^2 = 25$.)

   a. $\sqrt{5}$
   b. $\sqrt{10}$
   c. $\sqrt{50}$
   d. $\sqrt{45}$
   e. $\sqrt{65}$
   f. $\sqrt{32}$
   g. $\sqrt{18}$
   h. $\sqrt{1000}$

6. Using the table at the end of this chapter find approximations of the following to the nearest hundredths.

   a. $\sqrt{97}$
   b. $\sqrt{50}$
   c. $\sqrt{75}$
   d. $\sqrt{9025}$
   e. $\sqrt{4096}$

7. How long a ladder is needed to reach the roof of a house 9 feet above the ground if the bottom of the ladder must be 4 feet away from the side of the house?

8. On the map of Problem 7, Exercises 19-2d, find the distance between:
   a. The railroad grade crossing and the home plate of the baseball diamond.
   b. The door of the church and the gasoline pump on the right.

9. BRAINBUSTER. Using mathematical signs and symbols and without changing the position of the digits, make a true statement: 2 9 6 7 = 17.
   (Hint: Use the symbol for square root.)

## 19-9. Summary.

A coordinate system in the plane is established by choosing a horizontal line (X-axis) and a vertical line (Y-axis) and marking off coordinates on each line so that:

1. The point of intersection is the point labeled 0 on both lines;
2. The same unit of distance is used on both lines.

Such a coordinate system enables us to represent the location of any point in the plane by a coordinate pair. The point of intersection of the axes is called the origin and has coordinates (0, 0).

The X-axis separates the plane into two regions called the upper half-plane and the lower half-plane. The Y-axis separates the plane into two regions called the right half-plane and the left half-plane. The two axes separate the plane into four regions called quadrants.

Points lying on the same vertical line have the same X-coordinate. Points lying on the same horizontal line have the same Y-coordinate.

The points (x, y) for which $y = bx$ lie on a line which passes through the origin. This line is called the graph of the equation $y = bx$. Such graphs enable us to do multiplication problems graphically.

The distance between two points which lie on the same horizontal line is the difference of their X-coordinates. The distance between two points lying on the same vertical line is the difference of their Y-coordinates. To express the distance between two points when they lie in other positions we need to know the Pythagorean Property.

The Pythagorean Property is: the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.

The square of the distance between two points A and B is given by:

$$(\text{difference of X-coordinates})^2 + (\text{difference of Y-coordinates})^2.$$

The positive number whose square is 5 is denoted by $\sqrt{5}$ (read: the square root of five.) Square roots of whole numbers from 1 to 100 are given to the nearest thousandth in the table on pages 354-355.

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## 19-10. Chapter Review.

### Exercises 19-10

Fill in the blanks in Problems 1 through 8, using answers from the following list:

vertical
upper
X
I
Y
horizontal
III
lower
IV
line
origin
X-axis
Y-axis
positive
negative
half-plane

1. The points $(6, 2)$ and $(^-5, 2)$ lie on the same _____ line.
2. The points $(^-5, 7)$ and $(^-5, ^-3)$ lie on the same _____ line.
3. Points in the _____ half-plane have Y-coordinates positive.
4. Points in the left half-plane have _____ coordinates _____.
5. The graph of the set of points $(x, y)$ for which $y > x$ is a _____.
6. The graph of the set of points $(x, y)$ for which $y = 3x$ is a _____ through the _____.
7. Except for one point, the graph in Problem 6 lies entirely in quadrants _____ and _____.
8. The graph of the set of points $(x, y)$ for which $y = {}^-2x$ lies in quadrants _____ and _____.

9. A certain point lying on the graph of $y = x^2$ has X-coordinate = 3. What is its Y-coordinate?

10. Without plotting, give the quadrant in which each point lies:

    a. $(7, {}^{-}5)$
    b. $({}^{-}3, {}^{-}7)$
    c. $(4, 5)$
    d. $({}^{-}8, 2)$
    e. $(2, {}^{-}4)$
    f. $({}^{-}2, {}^{-}4)$

11. Find the distance between $(2, 3)$ and $(3, 2)$.

12. Find the distance between $({}^{-}16, {}^{-}5)$ and $(24, 4)$.

13. Using the table at the end of the chapter, find decimal approximations to the following square roots to the nearest thousandth:

    a. $\sqrt{43}$
    b. $\sqrt{27}$
    c. $\sqrt{81}$
    d. $\sqrt{121}$
    e. $\sqrt{4489}$

14. In a certain rectangle, the diagonal is only 1 inch longer than the longer side. If the longer side is one foot in length, how long is the shorter side?

---

## 19-11. Cumulative Review.

### Exercises 19-11

1. The vertices of a quadrilateral are the points $({}^{-}4, 3)$, $(0, 0)$, $(4, 0)$, and $(0, 3)$. Plot these points. Find the lengths of its sides. What kind of quadrilateral is it?

2. Find the value of

   a. $^{-}1 + 2 + {}^{-}(\frac{1}{2}) + \frac{1}{5}$

   b. $\frac{2}{3} \cdot {}^{-}(\frac{2}{7})$

   c. $^{-}(\frac{1}{2}) \div {}^{-}(\frac{1}{4})$

3. Arrange the following set of numbers in order, from smallest to largest.

   $\{^{-}10,\ 0.001,\ ^{-}1.1,\ 0.01,\ 0.009,\ ^{-}1.099\}$

4. Which is larger? $\frac{9}{12}$ or $\frac{35}{44}$

5. Subtract $^{-}20$ from 30, then divide the result by 5, then multiply this quotient by 2, and then divide by $^{-}20$.

6. If a number is represented by $\frac{2n - 7}{3}$, find its value when

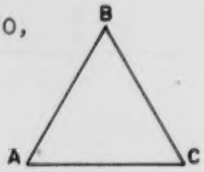
   a. $n = 0$

   b. $n = {}^{-}1$

   c. $n = 5$

7. An oil storage tank is in the shape of a rectangular prism, 15 ft. long, 12 ft. wide, and 9 ft. high. How many loads of oil must a truck carry to fill the tank? The truck is a circular cylinder which is 10 feet long, and whose radius is 3 feet. (Use $\pi = 3.14$.)

8. In triangle ABC, AB = BC and $m\angle ABC = 70$,



   a. find $m\angle C$

   b. find $m\angle A$

   c. what kind of triangle is $\triangle ABC$?

9. A circular cylinder is 3 ft. high and the diameter of its base is 5 ft. (Use $\pi = 3.14$.)

   a. How many cubic feet are in its volume?

   b. How many square feet of total surface area does it have?

# T A B L E

## SQUARES AND SQUARE ROOTS OF NUMBERS

| No. | Squares | Square roots | No. | Squares | Square roots |
|---|---|---|---|---|---|
| 1 | 1 | 1.000 | 36 | 1,296 | 6.000 |
| 2 | 4 | 1.414 | 37 | 1,369 | 6.083 |
| 3 | 9 | 1.732 | 38 | 1,444 | 6.164 |
| 4 | 16 | 2.000 | 39 | 1,521 | 6.245 |
| 5 | 25 | 2.236 | 40 | 1,600 | 6.325 |
| 6 | 36 | 2.449 | 41 | 1,681 | 6.403 |
| 7 | 49 | 2.646 | 42 | 1,764 | 6.481 |
| 8 | 64 | 2.828 | 43 | 1,849 | 6.557 |
| 9 | 81 | 3.000 | 44 | 1,936 | 6.633 |
| 10 | 100 | 3.162 | 45 | 2,025 | 6.708 |
| 11 | 121 | 3.317 | 46 | 2,116 | 6.782 |
| 12 | 144 | 3.464 | 47 | 2,209 | 6.856 |
| 13 | 169 | 3.606 | 48 | 2,304 | 6.928 |
| 14 | 196 | 3.742 | 49 | 2,401 | 7.000 |
| 15 | 225 | 3.873 | 50 | 2,500 | 7.071 |
| 16 | 256 | 4.000 | 51 | 2,601 | 7.141 |
| 17 | 289 | 4.123 | 52 | 2,704 | 7.211 |
| 18 | 324 | 4.243 | 53 | 2,809 | 7.280 |
| 19 | 361 | 4.359 | 54 | 2,916 | 7.348 |
| 20 | 400 | 4.472 | 55 | 3,025 | 7.416 |
| 21 | 441 | 4.583 | 56 | 3,136 | 7.483 |
| 22 | 484 | 4.690 | 57 | 3,249 | 7.550 |
| 23 | 529 | 4.796 | 58 | 3,364 | 7.616 |
| 24 | 576 | 4.899 | 59 | 3,481 | 7.681 |
| 25 | 625 | 5.000 | 60 | 3,600 | 7.746 |
| 26 | 676 | 5.099 | 61 | 3,721 | 7.810 |
| 27 | 729 | 5.196 | 62 | 3,844 | 7.874 |
| 28 | 784 | 5.292 | 63 | 3,969 | 7.937 |
| 29 | 841 | 5.385 | 64 | 4,096 | 8.000 |
| 30 | 900 | 5.477 | 65 | 4,225 | 8.062 |
| 31 | 961 | 5.568 | 66 | 4,356 | 8.124 |
| 32 | 1,024 | 5.657 | 67 | 4,489 | 8.185 |
| 33 | 1,089 | 5.745 | 68 | 4,624 | 8.246 |
| 34 | 1,156 | 5.831 | 69 | 4,761 | 8.307 |
| 35 | 1,225 | 5.916 | 70 | 4,900 | 8.367 |

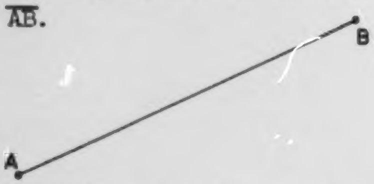
| No. | Squares | Square roots | No. | Squares | Square roots |
|---|---|---|---|---|---|
| 71 | 5,041 | 8.426 | 86 | 7,396 | 9.274 |
| 72 | 5,184 | 8.485 | 87 | 7,569 | 9.327 |
| 73 | 5,329 | 8.544 | 88 | 7,744 | 9.381 |
| 74 | 5,476 | 8.602 | 89 | 7,921 | 9.434 |
| 75 | 5,625 | 8.660 | 90 | 8,100 | 9.487 |
| 76 | 5,776 | 8.718 | 91 | 8,281 | 9.539 |
| 77 | 5,929 | 8.775 | 92 | 8,464 | 9.592 |
| 78 | 6,084 | 8.832 | 93 | 8,649 | 9.644 |
| 79 | 6,241 | 8.888 | 94 | 8,836 | 9.695 |
| 80 | 6,400 | 8.944 | 95 | 9,025 | 9.747 |
| 81 | 6,561 | 9.000 | 96 | 9,216 | 9.798 |
| 82 | 6,724 | 9.055 | 97 | 9,409 | 9.849 |
| 83 | 6,889 | 9.110 | 98 | 9,604 | 9.899 |
| 84 | 7,056 | 9.165 | 99 | 9,801 | 9.950 |
| 85 | 7,225 | 9.220 | 100 | 10,000 | 10.000 |

# Chapter 20

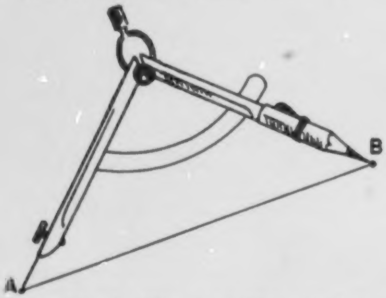
# REAL NUMBERS

## 20-1. Real Numbers as Points on the Number Line.

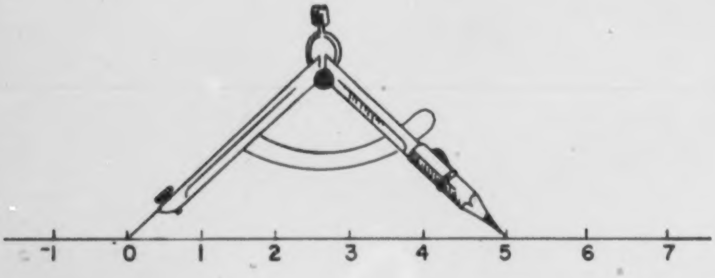
In your earlier work you learned to use the number line to give the lengths of segments. Now recall how this is done. Here is a segment $\overline{AB}$.



To find the length of this segment, open a compass and place the needle at A and the pencil point at B.

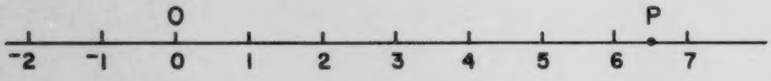


Then move the compass so that the needle and pencil point fall on the number line with the needle at the origin and the pencil point to the right of the origin.



The number corresponding to the point at which the pencil point falls tells us the length of the segment $\overline{AB}$. In the case shown that length is 5.

In particular if we choose a point P on the number line then the number corresponding to P is the length of the segment $\overline{OP}$.



By now this must seem very obvious but it leads to a very important conclusion. If every segment is to have a length, then for every point on the number line there must be a corresponding number. (Perhaps we should only say that every point to the right of the origin must have a corresponding number. But then if every number is to have an opposite, the points to the left of zero must have corresponding numbers too.)

We are going to agree that every point on the number line represents a number. (Probably many of you have been operating under this assumption already. If you have, so much the better.) The set of numbers represented by the entire set of points on the number line is called the set of real numbers.

The only kind of numbers we have met so far are the rational numbers (numbers that can be written as fractions $\frac{a}{b}$ where a and b are whole numbers and $b \neq 0$ ). When we locate our rational numbers on the number line does every point have a rational number that corresponds to it? Are the set of real numbers and the set of rational numbers the same? For many years it was considered obvious that they were the same. We shall find the answer to these questions later in this chapter.

Real numbers can be added and subtracted geometrically by the method we used in earlier chapters. For example if x and y are represented by the points indicated in Figure 20-1a, then

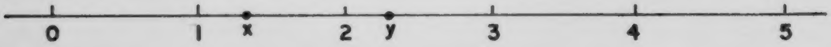


Figure 20-1a

$x + y$ is found as shown in Figure 20-1b.

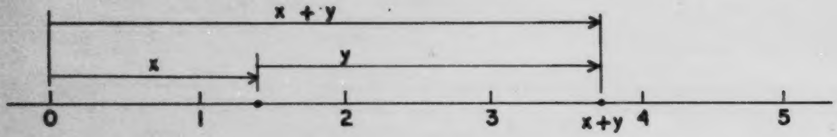


Figure 20-1b

The difference $x - y$ is shown in Figure 20-1c.

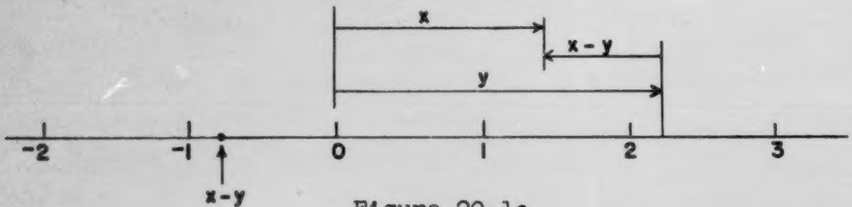


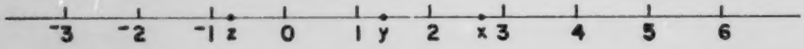
Figure 20-1c

The set of real numbers is closed under addition and subtraction.

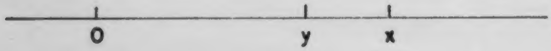
There is also a geometrical method for multiplying and dividing real numbers, but this method is rather complicated. However, the set of real numbers is also closed under multiplication and division (except for division by zero). The commutative, associative and distributive properties also hold for the operations of addition and multiplication on the real numbers.

### Exercises 20-1

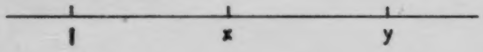
1. On the number line below locate the points corresponding to $x + y$, $y - x$, $2x$, $x + z$, $y - z$, $z - y$, $3z$.



2. On the number line below locate $x + y$ and $x - y$.



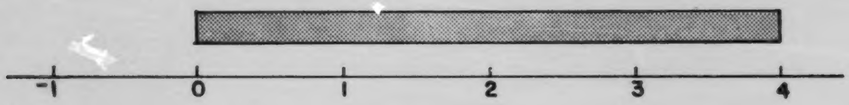
3. On the number line below can you locate $x + y$ or $x - y$? Explain.



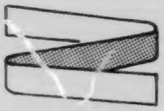
## 20-2. Locating Numbers on the Number Line.

In an earlier chapter we learned how to locate on the number line numbers expressed in the form $\frac{\text{"}a\text{"}}{b}$ where $a$ and $b$ are whole numbers. Let us recall how this can be done.

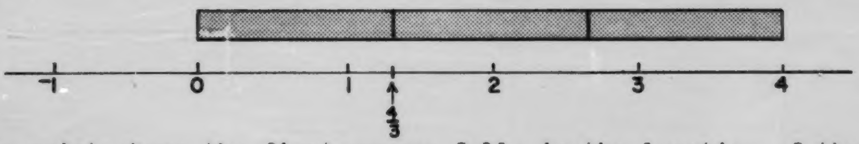
To locate $\frac{4}{3}$ on the number line we take a ribbon or strip of paper 4 units in length,



fold it into three equal parts,

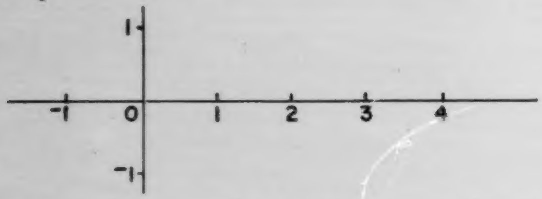


crease it at the folds and lay it off on the number line.

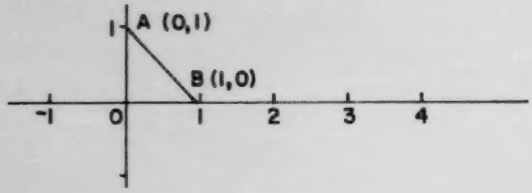


The point where the first crease falls is the location of the number $\frac{4}{3}$.

Here is a somewhat different problem. In the last chapter we found a number $n$ whose square is 2 (that is $n \cdot n = 2$). This number is called $\sqrt{2}$. Let us try to locate this number on the number line. First we construct a coordinate plane using our number line as X-axis.



Next, draw the segment joining the points A (0,1) and B (1,0). Now these points A and B together with the origin (0,0) form the vertices of a right triangle.



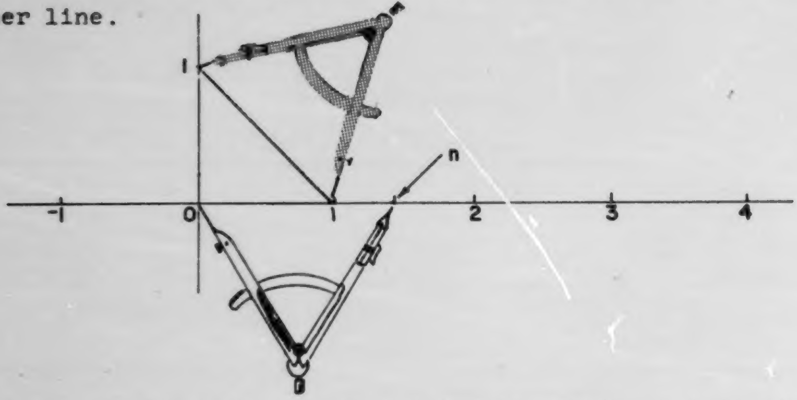
By the Pythagorean Theorem the length, $n$, of the segment $\overline{AB}$ satisfies the condition that

$$n \cdot n = 1 \cdot 1 + 1 \cdot 1$$

or

$$n \cdot n = 2.$$

Now we may use the compass to transfer this length $n$ onto the number line.



The point at which the pencil point falls is the point which corresponds to $\sqrt{2}$.

## Exercises 20-2

1. From the relation $1^2 + 2^2 = 5$, show how to locate $\sqrt{5}$ on the number line.

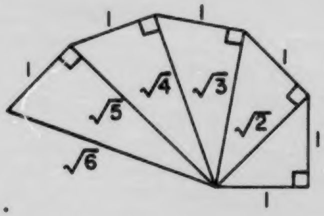
2. From the relation $2^2 + 3^2 = 13$, show how to locate $\sqrt{13}$ on the number line.

3. The relation $1 + 5 = 6$ could be rewritten as $1^2 + (\sqrt{5})^2 = 6$. Use this relation together with the result of Problem 1 to locate $\sqrt{6}$ on the number line.

4. Use the geometrical method of adding real numbers to locate $\sqrt{5} + \sqrt{13}$.

5. Use the geometrical method of subtraction to locate $\sqrt{5} - \sqrt{13}$.

6. The picture at the right has been made using right angles and segments of length 1. Use the Pythagorean Theorem to check that the lengths of the other segments are as indicated. Make a careful copy of this picture on your paper and continue the process until you have drawn a length to represent $\sqrt{10}$.



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## 20-3. Irrational Numbers.

In the last section we located on the number line the number $\sqrt{2}$. This is a number $n$ for which $n^2 = 2$ or $n \cdot n = 2$. In this section we shall try to find out whether this number $n$ is rational. If $n$ is rational, then $n$ can be expressed as a fraction in simplest form. This means that there are whole numbers $a$ and $b$ with

$$n = \frac{a}{b}$$

and with $a$ and $b$ having no factors in common. (This is what we mean by simplest form.)

Now if $n = \frac{a}{b}$ and $n \cdot n = 2$ this means that

$$\frac{a}{b} \cdot \frac{a}{b} = 2 \qquad \text{or} \qquad \frac{a \cdot a}{b \cdot b} = 2 \qquad \text{or} \qquad a \cdot a = 2 \cdot b \cdot b$$

or finally

$$a^2 = 2b^2$$

Now we see that if this number $n$ is rational, then there are whole numbers $a$ and $b$ which meet the following requirements:

(1) $a^2 = 2b^2$ ;

(2) $a$ and $b$ have no factors in common.

If there are no numbers $a$ and $b$ satisfying these requirements then $n$ is a kind of number which is quite new to us; that is, $n$ is a number which is not rational.

It might be very difficult to find whether there are any whole numbers $a$ and $b$ which meet the requirements (1) and (2). We certainly cannot check all possible pairs of whole numbers. But we shall find the answer by checking whether $a$ and $b$ could be even or odd. There are just these four possible cases:

| | |
|---|---|
| I ($a$ even, $b$ even) | III ($a$ even, $b$ odd) |
| II ($a$ odd, $b$ even) | IV ($a$ odd, $b$ odd) |

We will check for each of these cases whether such numbers could possibly meet requirements (1) and (2).

Case I ($a$ even, $b$ even). Since $a$ and $b$ are both even, they are both multiples of 2. This means that $a$ and $b$ will not meet requirement (2) since they have 2 as a common factor. Case I is ruled out.

Cases II or IV ($a$ odd, $b$ even) or ($a$ odd, $b$ odd). In either of these cases $a$ is odd. Now when we write $a^2$ as $a \cdot a$ we see that $a^2$ is the product of two odd factors. This tells us that $a^2$ is an odd number. (Remember that the product of two odd numbers is always an odd number.) Now consider the whole numbers $a^2$ and $2b^2$.

$a^2$ ↑ just shown to be odd

$2b^2$ ↑ a multiple of 2 and therefore even

The numbers $a^2$ and $2b^2$ cannot be equal since one of these numbers is odd and the other even. Requirement (1) $a^2 = 2b^2$ is not met. Cases II and IV are ruled out.

Case III (a even, b odd). This is the only case not yet ruled out. In this case, since a is even, we see that $a^2$ (which is $a \cdot a$) is the product of two even factors so that $a^2$ is a multiple of 4. (Do you see why the product of two even factors is always a multiple of 4?) And the number $b^2$ is the product of two odd factors so that $b^2$ is odd. Now consider the numbers $a^2$ and $2b^2$.

$a^2$ ↑ just seen to be a multiple of 4

$2b^2$ ↑ 2 times an odd number, therefore not a multiple of 4

The numbers $a^2$ and $2b^2$ cannot be equal since one of these numbers is a multiple of 4 and the other is not. Requirement (1) $a^2 = 2b^2$ is not met. Case III is ruled out.

All four cases have been ruled out! This means that there are no whole numbers a and b satisfying requirements (1) and (2). As we saw at the beginning of this section, this means that this number n, or $\sqrt{2}$, is a kind of number which is quite new to us. It is a number which is not rational. Real numbers which are not rational are called irrational numbers.

So far we have only seen one example of an irrational number, namely $\sqrt{2}$. From this one example many other irrational numbers may be found. It is easy to show (although we shall not do it here) that the sum of a rational number and an irrational number is always irrational. Thus $3 + \sqrt{2}$ and

$\sqrt{2} + \frac{4}{7}$, etc., are irrational. Similarly, the product of an irrational and a non-zero rational is irrational. Thus $5 \cdot \sqrt{2}$, $^{-}2\sqrt{2}$ and $\frac{3}{4} \cdot \sqrt{2}$ are irrational. Some other irrational numbers are $\sqrt{5}$, $\sqrt{7}$, $\pi$. In fact in a certain sense there are many more irrational numbers than there are rational numbers. This may seem like a very astonishing statement to you. Perhaps you will ask, "There are infinitely many rational numbers, how can there be more than that?" The sense in which there are more irrationals than rationals is this: If you try to set up a one-to-one correspondence between rationals and irrationals, no matter how you do it, when the rationals are all used up there will still be infinitely many irrationals left over.

We see that when we represented rational numbers as points on the number line, there were many "holes" or points to which no numbers were assigned. When the real numbers are represented as points on the number line there are no "holes". There is a number corresponding to every point. We therefore say that the real number system is *complete*.

A thorough study of the real numbers involves many advanced ideas which are introduced in college mathematics. In this chapter we give only an introduction to the simpler ideas and we shall attempt no further proofs.

## Exercises 20-3

1. Draw a number line. Include numbers from $^{-}5$ to $5$. Use the letter A for the point 0 and the letter B for the point 1. At B draw a segment perpendicular to the number line and 1 unit in length and call it $\overline{BP}$. Draw $\overline{AP}$. What is the measure of segment $\overline{AP}$ ?

2. Use the drawing for Problem 1, and locate on the number line points which correspond to $\sqrt{2}$ and $^{-}(\sqrt{2})$. Label the points.

3. On the number line locate and label the points which correspond to the following:

   a. $\sqrt{2} + \sqrt{2}$

   b. $3\sqrt{2}$

   c. $^{-}3\sqrt{2}$

   d. $\sqrt{4}$

   e. $\sqrt{2} + 2$

   f. $\sqrt{2} - 2$

   g. $\frac{1}{2}\sqrt{2}$

4. Which of the points that you have labeled correspond to irrational numbers?

---

## *20-4. Irrational Numbers and Infinite Decimals.

In earlier chapters we found how to represent any rational number as a repeating decimal. Some examples are

$$\frac{1}{3} = .333\ \ldots \qquad \text{or} \qquad \frac{1}{3} = .3\overline{3}.$$

$$\frac{2}{7} = .28571428571 42\ \ldots \qquad \text{or} \qquad \frac{2}{7} = .285714\overline{285714}.$$

Remember that the bars mean that continuing the process of division will just result in a repetition of the same digits over and over again without end. For this reason we say that these repeating decimals are "infinite decimals".

You know that every rational number can be represented as a repeating decimal. The converse is also true. That is, every repeating decimal represents a rational number. We show how to find that number in the following example. Consider the number

$$.\overline{162} \qquad \text{or} \qquad .162162162\ \ldots\ .$$

Now

$$1000 \cdot (.\overline{162}) = 162.162162\ \ldots$$

or

$$1000 \cdot (.\overline{162}) = 162.\overline{162}.$$

(We multiplied by $10^3$ because the repeating block was 3 digits long.) Now we perform the subtraction below:

$$\begin{array}{r} 1000 \cdot (.\overline{162}) = 162.\overline{162} \\ 1 \cdot (.\overline{162}) = \quad .\overline{162} \\ \hline 1000 \cdot (.\overline{162}) - 1 \cdot (.\overline{162}) = 162 \end{array}$$

Then $1000 \cdot (.\overline{162}) - 1 \cdot (.\overline{162}) = 999 \cdot (.\overline{162})$

so that $999 \cdot (.\overline{162}) = 162.$

Therefore

$$.\overline{162} = \frac{162}{999} = \frac{6}{37} .$$

You may check this result by starting with $\frac{6}{37}$ and finding its repeating decimal representation.

## Exercises 20-4a

1. Write the products.

a. $10 \times 0.999\overline{9}$

b. $100 \times 3.12\overline{12}$

c. $1{,}000 \times 0.035\overline{035}$

d. $10 \times 16.66\overline{6}$

e. $10 \times 0.0044\overline{4}$

f. $1{,}000 \times 0.61345\overline{345}$

g. $100 \times 8.0315\overline{15}$

h. $100 \times 312.899\overline{9}$

i. $10 \times 312.899\overline{9}$

j. $10{,}000 \times 6.01230\overline{123}$

2. Subtract in each of the following.

a. $\begin{array}{r} 3128.99\overline{9} \\ 312.89\overline{9} \\ \hline \end{array}$

b. $\begin{array}{r} 9.99\overline{9} \\ 0.99\overline{9} \\ \hline \end{array}$

c. $\begin{array}{r} 162.162\overline{162} \\ 0.162\overline{162} \\ \hline \end{array}$

d. $\begin{array}{r} 301.0101\overline{01} \\ 3.0101\overline{01} \\ \hline \end{array}$

e. $\begin{array}{r} 1.23333\overline{3} \\ 0.12333\overline{3} \\ \hline \end{array}$

f. $\begin{array}{r} 354.54\overline{54} \\ 3.54\overline{54} \\ \hline \end{array}$

g. $\begin{array}{r} 27075.075\overline{075} \\ 27.075\overline{075} \\ \hline \end{array}$

h. $\begin{array}{r} 416.4777\overline{7} \\ 41.6477\overline{7} \\ \hline \end{array}$

3. Find a fraction name for each of the following rational numbers written as repeating decimals.

a. $0.55\overline{5}$

b. $0.73\overline{73}$

c. $0.901\overline{901}$

*d. $3.023\overline{33}$

In the earlier sections of this chapter we learned how to locate the number $\sqrt{2}$ on the number line and we learned that this number $\sqrt{2}$ is not rational. Now we shall try to find some kind of decimal representation for $\sqrt{2}$.

When we located $\sqrt{2}$ on the number line we found that it fell at a point between 1 and 2.

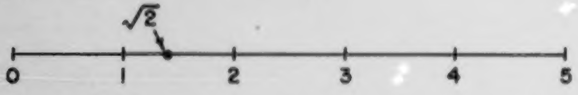


Figure 20-4a

We can see that $\sqrt{2}$ must be so situated even without the picture if we remember that $\sqrt{2}$ is a number whose square is 2. The number 1 has a square less than 2 while the number 2 has a square greater than 2.

If in Figure 20-4a we magnify 10 times the interval in which $\sqrt{2}$ lies, we have:

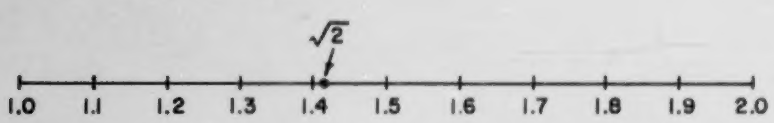


Figure 20-4b

with the points corresponding to "tenths" indicated. The number $\sqrt{2}$ lies between 1.4 and 1.5 since $(1.4)^2 = 1.96$ which is less than 2, while $(1.5)^2 = 2.25$ which is greater than 2.

Now, if the interval in which $\sqrt{2}$ lies in Figure 20-4b is magnified 10 times and the points corresponding to "hundredths" are indicated we have:

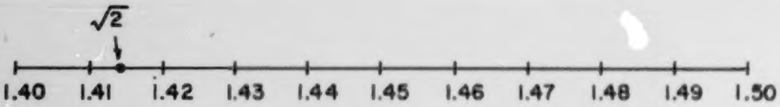


Figure 20-4c

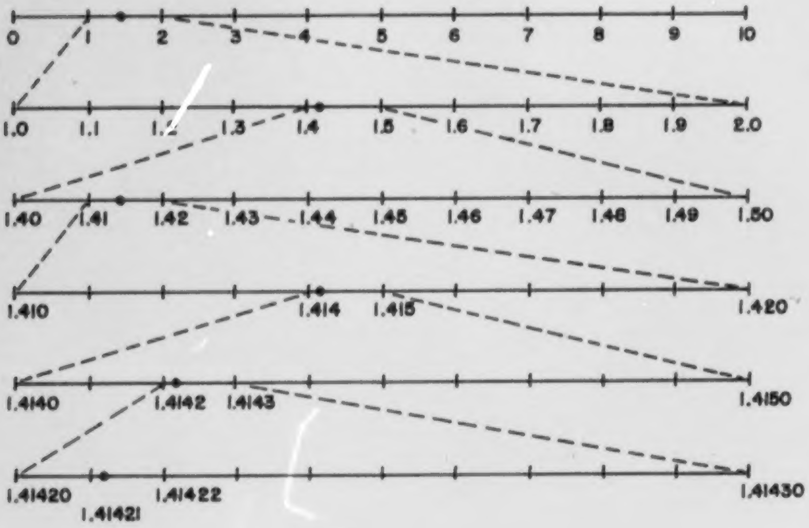
The number $\sqrt{2}$ lies in the depicted interval since

$$(1.41)^2 = 1.9881 \qquad \text{while} \qquad (1.42)^2 = 2.0164.$$

It can be seen that

$$1.41 < \sqrt{2} < 1.42.$$

We can repeat this process as long as our patience holds out. We show below the steps performed above together with a few additional steps. Dotted lines are used to show which segments are being magnified.


0
1
2
3
4
5
6
7
8
9
10
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
1.40
1.41
1.42
1.43
1.44
1.45
1.46
1.47
1.48
1.49
1.50
1.410
1.414
1.415
1.420
1.4140
1.4142
1.4143
1.4150
1.41420
1.41421
1.41422
1.41430

We see that

$$1.41421 < \sqrt{2} < 1.41422$$

since

$$(1.41421)^2 = 1.9999899241$$

while

$$(1.41422)^2 = 2.0000182084.$$

Continuing this process indefinitely would yield an infinite decimal representing the number $\sqrt{2}$. This infinite decimal has the property that if it is chopped off after any number of digits, the resulting number has a square less than 2, while if the last digit in this chopped-off decimal is increased by 1, the resulting number has a square greater than 2.

We can be sure that this infinite decimal is not a repeating decimal since $\sqrt{2}$ is not rational. Remember that repeating decimals always represent rational numbers. Infinite decimals which are not repeating represent irrational numbers.

The rule for finding the digits in the infinite decimal representing $\sqrt{2}$ is a rather complicated one. Here is a non-repeating infinite decimal which has a rather simple rule for finding the digits.

$$.737733777333777733337\ldots$$

You have probably discovered the rule: one 7, one 3, two 7's, two 3's, three 7's, three 3's, etc. We can see that this is not a repeating decimal because the blocks of consecutive 3's (or 7's) get longer and longer with no longest block. In a repeating decimal there would have to be a longest block of 3's which would repeat over and over.

## Exercises 20-4b

1. Arrange each group of decimals in the order in which the points to which they correspond occur on the number line. List first the point farthest to the left.

   a. 1.379 1.493 1.385 5.468 1.372

   b. $^{-}9.426$ $^{-}2.765$ $^{-}2.761$ $^{-}5.630$ $^{-}2.763$

   c. 0.15475 0.15467 0.15463 $^{-}0.15475$ 0.15598

2. In Problem 1(c) which points lie on the following segments:
   - a. the segment with endpoints 1 and 2?
   - b. the segment with endpoints 0 and 1?
   - c. the segment with endpoints 0.1 and 0.2?
   - d. the segment with endpoints 0.15 and 0.16?
   - e. the segment with endpoints 0.154 and 0.155?

3. Draw a 10 centimeter segment; label the endpoints 0 and 1, and divide the segment into tenths. Mark and label the following points:
   - a. 0.23
   - b. 0.49
   - c. 0.80
   - d. 0.6
   - e. 0.08
   - f. 0.95

4. Which of the following are:
   - (1) rational numbers,
   - (2) irrational numbers,
   - (3) not designated clearly enough for you to be able to tell.

   - a. $0.231\overline{231}$
   - b. $0.23123112311123\ldots$
   - c. $0.7500\overline{0}$
   - d. $0.78342\overline{342}$
   - e. $0.959559555955559\ldots$
   - f. $0.3478512\ldots$
   - g. $5\sqrt{2}$
   - h. $\sqrt{2} - 5$
   - i. $\frac{58}{11}$
   - j. $\frac{\pi}{2}$

5. Express (a), (b), and (c) as decimals to six places.
   - a. $(1.731)^2$
   - b. $(1.732)^2$
   - c. $(1.733)^2$
   - d. Find the difference between your answer for (a) and the number 3; find the difference between your answer for (b) and the number 3; find the difference between your answer for (c) and the number 3.
   - e. To the nearest thousandth what is the best decimal expression for $\sqrt{3}$?

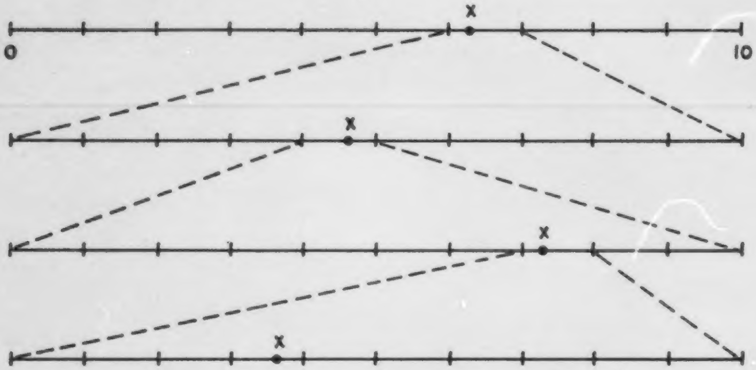
Which of the numbers suggested is the better approximation of the following irrational numbers?

6. $\sqrt{3}$ : 1.73 or 1.74

7. $\sqrt{15}$ : 3.87 or 3.88

8. $\sqrt{637}$ : 25.2 or 25.3

9. In the picture below, a certain number x is shown in repeatedly magnified pictures (as was done for $\sqrt{2}$ in this section). Give as many decimal places as you can in the decimal representation of this number.



10. Draw a repeatedly magnified picture representation for the number $\pi = 3.141 \ldots$ .

## 20-5. Properties of Number Systems.

In the course of our work we have observed the following properties hold true in the rational number system.

<table>
<tr><th>Addition</th><th>Multiplication</th></tr>
<tr><td>Closure Property of Addition</td><td>Closure Property of Multiplication</td></tr>
<tr><td>Commutative Property<br>$a + b = b + a$</td><td>Commutative Property<br>$a \cdot b = b \cdot a$</td></tr>
<tr><td>Associative Property<br>$(a + b) + c = a + (b + c)$</td><td>Associative Property<br>$(a \cdot b) \cdot c = a \cdot (b \cdot c)$</td></tr>
<tr><td>Identity Property of 0<br>$a + 0 = 0 + a = a$</td><td>Identity Property of 1<br>$a \cdot 1 = 1 \cdot a = a$</td></tr>
<tr><td>Inverse Property<br>$a + (-a) = (-a) + a = 0$</td><td>Inverse Property<br>$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$<br>(for $a \neq 0$)</td></tr>
<tr><td colspan="2">Distributive Property<br>$a \cdot (b + c) = a \cdot b + a \cdot c$</td></tr>
<tr><td></td><td>Multiplication Property of 0<br>$a \cdot 0 = 0 \cdot a = 0$</td></tr>
</table>

Order Property

For any numbers $a$ and $b$, exactly one of the following is true:

$a < b$ $\qquad$ $a = b$ $\qquad$ $b < a$

All of these properties also hold true in the real number system and in addition the real number system has this property not enjoyed by the rational number system.

> Completeness: For every point on the number line there is a corresponding real number.

An equivalent but less geometrical statement of this completeness property would be:

> Completeness: Every infinite decimal represents a real number.

During our course we have studied several number systems besides the rational and real number systems. For all of these number systems the closure, commutative and associative properties of addition and multiplication hold true as do the distributive and order properties and the identity property of 1. The remaining properties hold for some of our systems and not others. In the table below we indicate by ✓ which properties hold in which systems.

SYSTEM

| PROPERTY | Counting Numbers | Whole Numbers | Integers | Non-Negative Rationals | Rational Numbers | Real Numbers |
|---|---|---|---|---|---|---|
| Identity Property of 0 |  | ✓ | ✓ | ✓ | ✓ | ✓ |
| Multiplicative Inverse |  |  |  | ✓ | ✓ | ✓ |
| Additive Inverse |  |  | ✓ |  | ✓ | ✓ |
| Multiplication Property of 0 |  | ✓ | ✓ | ✓ | ✓ | ✓ |

We see that the real number system and the rational number system both possess all these properties, but the real number system has the completeness property as well.

## Exercises 20-5

1. The set of integers is contained in which of the following sets of numbers?
   a. Counting numbers
   b. Whole numbers
   c. Non-negative rationals
   d. Rationals
   e. Reals
2. Given the number 1, what is the next larger:
   a. Counting number
   b. Whole number
   c. Integer
   d. Rational number
3. How many of the following numbers are there between $^{-}3$ and 6?
   a. Counting numbers
   b. Whole numbers
   c. Integers
   d. Rationals
   e. Reals
4. Given the number 1, what is the next smaller:
   a. Counting number
   b. Whole number
   c. Integer
   d. Rational number
5. Is the set of negative real numbers closed under the operations of:
   a. Addition
   b. Subtraction
   c. Multiplication
   d. Division

## 20-6. Summary.

1. Every point on the number line represents a real number.
2. Some real numbers are not rational.
3. Real numbers which are not rational are called irrational.
4. $\sqrt{2}$ is irrational.
5. Every infinite decimal represents a real number.
6. Numbers represented by repeating decimals are rational.
7. Numbers represented by decimals which do not repeat are irrational.

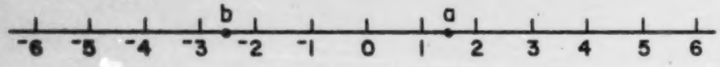
## 20-7. Chapter Review.

### Exercises 20-7

1. Which of the properties that are listed for the real numbers hold for
   - a. the set of even integers?
   - b. the set of odd integers?
2. Which of the following numbers are rational and which are irrational?

   The number of units in:
   - a. the circumference of a circle whose radius is $\frac{1}{2}$ unit.
   - b. the area of a square whose sides are one unit long.
   - c. the hypotenuse of a right triangle whose sides are 5 and 12 units long.
   - d. the area of a square whose sides have length $\sqrt{3}$ units.
   - e. the volume of a cylinder whose height is 2 units and whose base has radius 1 unit.
   - f. the area of a right triangle with hypotenuse of length 2 units and equal sides.

3. Locate $a + b$ and $a - b$ on the number line below:



4. The irrational number $\pi$ to eight places is $3.14159265\ldots$ An approximation which the Babylonians used for $\pi$ is the interesting ratio $\frac{355}{113}$. How good an approximation is this? Is it as good as $\frac{22}{7}$?

*5. What rational number is represented by $.63\overline{63}$?

---

## 20-8. Cumulative Review.

### Exercises 20-8

1. Sketch a number line and mark the points associated with the numbers:

$$3, \quad {}^{-}(\tfrac{5}{4}), \quad {}^{-}(\tfrac{3}{3}), \quad \tfrac{9}{2}, \quad {}^{-}(\tfrac{4}{3}), \quad \tfrac{0}{3}$$

2. Perform the indicated operations and simplify:
   - a. $\frac{2}{3} \times \frac{3}{4}$
   - b. $\frac{2}{3} + \frac{3}{4}$
   - c. $\frac{2}{3} - \frac{3}{4}$
   - d. $\frac{2}{3} \div \frac{3}{4}$
   - e. $\left(\frac{1}{2} - \frac{1}{6}\right) + \frac{1}{9}$

3. Write as a fraction in simplest form.
   - a. 0.6
   - b. 12.6
   - c. 0.035

4. a. Find the product of 60.5 and 1237.1.
   b. Find the quotient $81.9 \div 0.78$.

5. Is $\frac{216}{126}$ equal to, greater than, or less than $\frac{284}{161}$?

6. **Simplify:**

a. $\frac{^-10}{^-2}$ d. $\frac{0}{^-6}$

b. $\frac{2}{10}$ e. $\frac{^-9}{^-3}$

c. $\frac{^-10}{2}$ f. $\frac{6}{^-18}$

7. Translate each of the following phrases into symbols:
   - a. The number x subtracted from 28.
   - b. Three times the number x.
   - c. The product of 7 and the sum of 5 and a number x.

8. a. The measure of an acute angle is greater than ______ and less than ______.
   b. The measure of an obtuse angle is greater than ______ and less than ______.
   c. The measure of a right angle is ______.

9. Find the volume of a closet $7\frac{1}{2}$ feet high if the area of the floor is $9\frac{1}{3}$ sq. ft.

10. Find the area of the polygon ABCD shown in Figure 1.

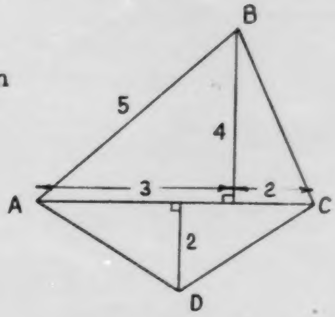


Figure 1

11. What is the area in Figure 2, which lies in the interior of the large circle but in the exterior of the two smaller circles?

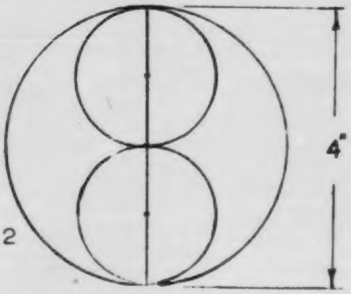


Figure 2

# Chapter 21

## SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM

### 21-1. Large Numbers and Scientific Notation.

Donald and David were playing a game with their younger sister Penny. Donald started it by boasting, "I know bigger numbers than you do." David replied, "I'll bet you don't." Donald started with "a hundred." David gained the advantage with "one million." After some thought Donald shouted "one billion"! That was as far as they could go, and Donald was just about to be declared the victor when little Penny spoke up with "one more than one billion." The boys argued for the rest of the afternoon about who was the winner.

Who do you think was the winner?
If you had been there, would you have won the game?
What is the largest number that you know?
Do you know how to read the following number?

3,141,592,653,589,793

Although we enter the commas from right to left, we read the numeral from left to right according to the following diagram:

| quadrillion | trillion | | | billion | | | million | | | thousand | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| one | hundred | ten | one | hundred | ten | one | hundred | ten | one | hundred | ten | one | hundred | ten | one |
| 3, | 1 | 4 | 1, | 5 | 9 | 2, | 6 | 5 | 3, | 5 | 8 | 9, | 7 | 9 | 3 |

We read this number as follows:

Three quadrillion,
one hundred forty-one trillion,
five hundred ninety-two billion,
six hundred fifty-three million,
five hundred eighty-nine thousand,
seven hundred ninety-three.

In reading such a numeral be careful not to use the word "and." If you say "five hundred and ninety-three thousand," there may be some misunderstanding. If "and" is associated with addition, the meaning is $500 + 93,000$. If "and" is interpreted as it is in ordinary English, the meaning is the two separate numbers, 500 and 93,000. Therefore, it is better to read 593,000 as "five hundred ninety-three thousand." By omitting "and" you avoid being misunderstood. The word "and" is usually used to mark the decimal point; for example, 500.093 is read "five hundred and ninety-three thousandths." This use of "and" does not cause confusion since 500.093 means $500 + .093$.

Actually, such numbers as 3,141,592,653,589,793 seldom occur. This does not mean that numbers of this size are not used, but merely that we rarely can count precisely enough to use such a number. Probably we would say that the number counted is about three quadrillion. The population of a city of over a million inhabitants might be given as 1,576,961. But this figure is just the sum of the various numbers compiled by the census takers. It is certain that the number changed while the census was being taken, but it is probable that 1,577,000 would be correct to the nearest thousand. For this reason there is no harm in rounding the original number to 1,577,000. For most purposes we say that the population of the city is "about one and one-half million." This can be written:

$$\text{Population of city} \approx 1,500,000.$$

Recall that the symbol $\approx$ is used to mean "is approximately equal to."

There are other ways of writing this number and there are some advantages in doing this. A hint of how this can be done is given by our statement "one and one-half million."

One million can be written: 1,000,000, or $(10 \times 10 \times 10 \times 10 \times 10 \times 10)$ or $10^6$. The notation $10 \times 10 \times 10 \times 10 \times 10 \times 10$ can be read "the product of six tens." The exponent 6 indicates the number of tens used as factors in the product. We can find the exponent by counting the number of zeros in the numeral 1,000,000. One and one-half million is the same as $1.5 \times 10^6$.

Let us write 2000 using an exponent. Since 2000 is

$$2 \times 1000,$$

then 2000 can be written, in the exponent form, as

$$2 \times 10^3.$$

## Exercises 21-1a

1. Write the following as decimal numerals.
   - a. one billion
   - b. one trillion
   - c. one quadrillion
   - d. one thousand
   - e. ten thousand
   - f. one hundred

2. Write the decimal numerals in Problem 1 in exponent form.

3. Write each of the following, using an exponent.
   - a. 7000
   - b. 50,000
   - c. 3,000,000
   - d. 14,000,000
   - e. 375,000,000
   - f. 480,000,000,000

4. Complete the following statements:
   - a. $7{,}500 = 75 \times \underline{\ ?\ }$
   - b. $8{,}760{,}000 = 876 \times \underline{\ ?\ }$
   - c. $83{,}000 = \underline{\ ?\ } \times 100$
   - d. $83{,}000 = \underline{\ ?\ } \times 1000$
   - e. $5{,}700{,}000 = 57 \times \underline{\ ?\ }$
   - f. $5{,}700{,}000 = \underline{\ ?\ } \times 10^2$
   - g. $5{,}700{,}000 = \underline{\quad} \times 10^5$
   - h. $5{,}700{,}000 = 570 \times \underline{\ ?\ }$
   - *i. $420 = \underline{\ ?\ } \times 100$
   - *j. $321 = \underline{\ ?\ } \times 10^2$

The number 1500 can be expressed in several ways:

$$150 \times 10 \text{ or,}$$
$$15 \times 100 \text{ or,}$$
$$1.5 \times 1000 \text{ or,}$$
$$1.5 \times 10^3.$$

Similarly, 325 can be written as

$$32.5 \times 10 \text{ or}$$
$$3.25 \times 10^2.$$

In each of these examples the last expression used is of the form:

(a number between 1 and 10) × (a power of 10).

In the first case it is $1.5 \times 10^3$, and in the second case it is $3.25 \times 10^2$.

A number is expressed in scientific notation if it is written as the product of a number between 1 and 10 and the proper power of ten, or just as a power of ten.

Both $1.7 \times 10^3$ and $10^4$ are written in scientific notation.

## Exercises 21-1b
(Class Discussion)

1. Express each of the following in the form of
   (a number between 1 and 10) × (a power of 10):
   Example: $4037 = 4.037 \times 10^3$.
   a. 76
   b. 859
   c. 7623
   d. 8463
   e. 76.4
   f. 483.5
   g. 841.2
   h. 9783.6
   i. 3412789.435

2. a. Is $15 \times 10^5$ in scientific notation? Why?
   b. Is $3.4 \times 10^7$ in scientific notation? Why?
   c. Is $12.0 \times 10^5$ in scientific notation? Why?

3. Express the following in scientific notation:
   a. 5687
   b. 14
   c. $3\frac{1}{2}$ million
   d. 27
   e. 613
   f. 205

4. Express the following in decimal notation.
   a. $3.7 \times 10^6$
   b. $4.7 \times 10^5$
   c. $5.721 \times 10^6$
   d. $1 \times 10^4$
   e. $1.6 \times 10^3$
   f. $8.3 \times 10^9$

5. Round each of the following to the nearest thousand.
   a. 6,400
   b. 6,600
   c. 675,123
   d. 6,500
   e. 6,501
   f. 6,499

## Exercises 21-1c

1. Write the following in scientific notation.
   a. 100
   b. 1000
   c. 10,000
   d. 687
   e. 6,000
   f. 7832
   g. $10 \times 10^5$
   h. $781 \times 10^7$
   i. $600 \times 10$
   j. 9,000,000,000

2. Write a numeral for each of the following in a form which does not use an exponent or indicate a product.
   - a. $10^3$
   - b. $10^5$
   - c. $5.83 \times 10^2$
   - d. $3 \times 10^4$
   - e. $6.3 \times 10^2$
   - f. $436 \times 10^6$
   - g. $10^9$
   - h. $17.3 \times 10^5$

3. Write the following, using words:
   - a. 783
   - b. 7,500,000
   - c. 63,007
   - d. 362.36
   - e. 284.63
   - f. 4.256

4. Round each of the following to the nearest hundred.
   - a. 645
   - b. 93
   - c. 1233
   - d. 70,863
   - e. 609
   - f. 362,449

5. Express the answers in Problem 4 in scientific notation.

---

## 21-2. Multiplying by Powers of Ten.

Scientific notation makes certain calculations easier and shorter. Suppose we want to find the value of the product:

$$100 \times 1000.$$

The first factor is the product of two tens, or $10^2$.
The second factor is the product of three tens, or $10^3$.
Then

$$\begin{aligned} 100 \times 1000 &= 10^2 \times 10^3 \\ &= (10 \times 10) \times (10 \times 10 \times 10) \\ &= 10 \times 10 \times 10 \times 10 \times 10 \\ &= 10^5 \end{aligned}$$

Let us see if we can find a short cut to the above method by looking at another example:

$1000 \times 10{,}000$ which in scientific notation is;

$$\begin{aligned} 10^3 \times 10^4 &= (10 \times 10 \times 10) \times (10 \times 10 \times 10 \times 10) \\ &= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 10^7 \end{aligned}$$

In the first example we find that

$$10^2 \times 10^3 = 10^5$$

and in the second example we find that

$$10^3 \times 10^4 = 10^7.$$

What will be the answer to the following example?

$$10^6 \times 10^5 = ?$$

Did you discover that the exponent 5 in the first example is the sum of the exponents, 2 and 3? In the second example, the exponent 7 is the sum of 3 and 4. In the third example, the exponent will be the sum of 6 and 5, or 11.

Find the product of 93,000,000 and 10,000. In scientific notation, this is:

$$\begin{aligned}(9.3 \times 10^7) \times (10^4) &= 9.3 \times (10^7 \times 10^4)\\ &= 9.3 \times 10^{11}\end{aligned}$$

## Exercises 21-2a

(Class Discussion)

1. Multiply and express your answer in scientific notation.
   - a. $10^7 \times 10^3$
   - b. $10^3 \times 10^2$
   - c. $10^4 \times 10^6$
   - d. $10^3 \times 10^8$
   - e. $10^2 \times 10^5$
   - f. $10^{14} \times 10^6$
   - g. $6 \times 10^7 \times 10^3$
   - h. $10^{13} \times 10^1 \times 10^5$
   - i. $1000 \times 1000$
   - j. $100 \times 10{,}000$
2. Multiply and express your answer in scientific notation.
   - a. $5 \times 16 \times 10^4$
   - b. $500 \times 10^5$
   - c. $2.3 \times 10^2 \times 10^3$
   - d. $8 \times 10^3 \times 5 \times 10^2$
   - e. $9 \times 100 \times 10^2 \times 6$
   - f. $4 \times 10^3 \times 25 \times 10$
   - g. $8 \times 10^2 \times 10^4 \times 50$
   - h. $3.7 \times 10^3 \times 10^5$
   - i. $80 \times 80 \times 80$
   - j. $8{,}000 \times 5{,}000$
   - k. $64{,}000 \times 105{,}000$
   - l. $1{,}200 \times 25 \times 150{,}000$
3. Sound travels in air about one-fifth of a mile in one second. Answer the following questions and express your answers in scientific notation.
   - a. How many miles will sound travel in air in one minute?

b. How many miles will sound travel in air in one hour?

4. A space ship travels twenty-five times as fast as sound. Answer the following questions and express your answers in scientific notation.
   a. How far will the space ship travel in 20 hours?
   b. How far will the space ship travel in 100 days?
   c. How far will the space ship travel in 2 years?
   d. Could it reach the sun in one year? (The distance from the earth to the sun is 93,000,000 miles.)

## Exercises 21-2b

1. Multiply, and express your answer in scientific notation:
   a. $6 \times 10^7 \times 10^3$
   b. $10^{13} \times 12 \times 10^4$
   c. $10^4 \times 3.5 \times 10^9$
   d. $300 \times 10^5$
   e. $10^2 \times 10^5 \times 7.63$
   f. $60 \times 60 \times 60$
   g. $9.3 \times 10^7 \times 10$

2. Write in scientific notation and multiply:
   a. 9,000,000 × 70,000
   b. 125 × 8,000,000
   c. 25,000 × 186,000
   d. 1100 × 5 × 200,000

3. The distance from the North Pole to the equator is about 10,000,000 meters.
   a. Express, in meters, the distance around the earth through the Poles in scientific notation.
   b. A meter is equal to one thousand millimeters. Express in scientific notation the distance in millimeters from the North Pole to the South Pole.
   c. One inch is about the same length as $2\frac{1}{2}$ centimeters. About how many centimeters will equal a distance of 40,000 feet?

4. The distance around the earth at the equator is about 25,000 miles. In one second, electricity travels a distance equal to about 8 times that around the earth

at the equator. About how far will electricity travel in 10 hours?

5. The earth's speed in its orbit around the sun is a little less than seventy thousand miles per hour. About how far does the earth travel in its yearly journey around the sun?

6. **BRAINBUSTER:** Suppose you have the task of making ten million marks on paper and you make two marks each second. Can you make 10,000,000 marks in (60 × 60 × 24 × 365) seconds?
How much time is this?

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## 21-3. Dividing by Powers of Ten.

You know $10^3 \times 10^2 = 10^5$, and $10^3 \times 10^4 = 10^7$. Can you tell in words how you multiply two numbers which are expressed as powers of 10?

The next exercise will help you discover a similar rule for dividing two numbers which are expressed as powers of 10.

### Exercises 21-3a

1. Let us divide $10^4$ by 10 and express the quotient as a power of 10:

$$\frac{10^4}{10} = \frac{10 \times 10 \times 10 \times 10}{10} = ?$$

2. Divide $\frac{10^6}{10^2} = \frac{10 \times 10 \times 10 \times 10 \times 10 \times 10}{10 \times 10} = ?$

3. $\frac{10^5}{10^3} = ?$

4. Have you discovered a "short-cut" method to get your answer other than the expanded form used in the first example? Did you notice that the exponent of 10 in the quotient is always the difference which results when you subtract the exponent of 10 in the denominator from the exponent of 10 in the numerator?

$$\frac{10^4}{10} = 10^{(4-1)} = 10^3$$

$$\frac{10^6}{10^2} = 10^{(6-2)} = 10^4$$

$$\frac{10^5}{10^3} = 10^{(5-3)} = 10^2$$

5. Divide $10^9$ by $10^3$.

$$\frac{10^9}{10^3} = ?$$

6. Express each answer as a power of 10.

$$\frac{10^4}{10} = ?$$

$$\frac{10^3}{10} = ?$$

$$\frac{10^2}{10} = ?$$

$$\frac{10}{10} = ?$$

It seems natural to answer the last question by writing:

$$\frac{10}{10} = 10^0$$

This answer follows the pattern of subtraction of exponents since

$$\frac{10}{10} = \frac{10^1}{10^1} = 10^{(1-1)} = 10^0.$$

It is also true that

$$\frac{10}{10} = 1$$

Hence if we wish to use zero as an exponent we must define $10^0$ as 1 .

Let us divide $10^0$ by 10.

$$\frac{10^0}{10} = \frac{1}{10}$$

If the pattern of exponents is to continue, we expect the next exponent to be 1 less than 0. You know that $0 - 1 = {}^{-}1$. It seems reasonable to define

$$10^{-1} \text{ as } \frac{1}{10} .$$

Now divide $10^{-1}$ by $10$.

$$\frac{1}{10} \div 10 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{10^2}$$

If the pattern of exponents is to continue, the new exponent should be 1 less than ${}^{-}1$, or ${}^{-}2$. Therefore we want to define

$$10^{-2} \text{ as } \frac{1}{10^2} \text{ or } \frac{1}{100} .$$

It is important to notice that the number $10^{-2}$ is not a negative number. It is a positive number, $\frac{1}{100}$.

Definition: If $n$ is a counting number

$$10^{-n} \text{ means } \frac{1}{10^n} .$$

This definition and the definition of $10^0$ enable us to write the powers of 10 as illustrated:

| $10^4$ | $10^3$ | $10^2$ | $10^1$ | $10^0$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ |
|---|---|---|---|---|---|---|---|
| 10,000 | 1000 | 100 | 10 | 1 | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1000}$ |

## Exercises 21-3b

1. Express each of the following using negative exponents:

a. $\frac{1}{10,000}$

b. $\frac{1}{1,000,000}$

c. $\frac{1}{10^7}$

d. $\frac{1}{10^2}$

e. $\frac{1}{10^9}$

f. $\frac{1}{100,000}$

g. $\frac{1}{10^3}$

h. $\frac{1}{10^{19}}$

i. .1

j. .0001

k. .001

l. .00001

2. Express each of the following in fractional form:
   - a. $10^{-3}$
   - b. $10^{-5}$
   - c. $10^{-7}$
   - d. $10^{-6}$
   - e. $10^{-27}$
   - f. $10^{-11}$
   - g. $10^{-256}$
   - h. $10^{-529}$

3. Express each of the following as a decimal numeral.
   - a. $10^{-2}$
   - b. $10^{-4}$
   - c. $10^{-1}$
   - d. $10^{-6}$

4. Express each of the following as a power of 10.
   - a. $\frac{100}{10}$
   - b. $\frac{10}{100}$
   - c. $\frac{10^3}{10^2}$
   - d. $\frac{10}{1000}$
   - e. $\frac{1000}{1000}$
   - f. $\frac{10^4}{10}$
   - g. $\frac{10}{10^4}$
   - h. $\frac{10^2}{1000}$
   - i. $\frac{1000}{1,000,000}$

You know that 0.4 means $\frac{4}{10}$, and that 0.004 means $\frac{4}{1000}$. What does 0.42 mean?

$$0.42 = \frac{4}{10} + \frac{2}{100} = \frac{40}{100} + \frac{2}{100} = \frac{42}{100} .$$

In a similar way, $0.6937 = \frac{6937}{10,000}$.

In Section 21-1, you used scientific notation to express large numbers. Now you are prepared to write small numbers in scientific notation.

Example 1. Write 0.004 in scientific notation.

$$0.004 = \frac{4}{1000} = \frac{4}{10^3} = 4 \times \frac{1}{10^3} = 4 \times 10^{-3}$$

Example 2. Write 0.00007 in scientific notation.

$$0.00007 = \frac{7}{100,000} = 7 \times \frac{1}{10^5} = 7 \times 10^{-5}$$

**Example 3.** Write 0.42 in scientific notation.

$$0.42 = \frac{42}{100} = \frac{4.2 \times 10}{100} = 4.2 \times \frac{10}{100} = 4.2 \times \frac{1}{10}$$
$$= 4.2 \times 10^{-1}$$

Our purpose in writing 42 as $4.2 \times 10$ is to secure a number between 1 and 10 for the first factor. Then $\frac{10}{100}$ is written in exponent form for the second factor.

**Example 4.** Write $0.16 \times 10^{-4}$ in scientific notation.

$$0.16 \times 10^{-4} = 0.16 \times \frac{1}{10^4} =$$

$$\frac{1.6}{10} \times \frac{1}{10^4} = 1.6 \times \frac{1}{10 \times 10^4} = 1.6 \times \frac{1}{10^5} = 1.6 \times 10^{-5}$$

Look at the preceding examples. Notice that .42 , .004, .00007, $0.16 \times 10^{-4}$ are smaller than 1. When they are written in scientific notation, the exponent of 10 is always negative. This is true for any number between 0 and 1.

When a number equal to or greater than 10 is expressed in scientific notation, the exponent of 10 is positive.

For 1 , or any number between 1 and 10 written in scientific notation, the exponent of 10 is zero.

**Examples:**

For a number between 0 and 1 , as $0.03 = 3 \times 10^{-2}$, the exponent of 10 is negative.

For a number between 1 and 10 , as $3.4 = 3.4 \times 10^0$, the exponent of 10 is zero.

For a number $\geq 10$ , as $13 = 1.3 \times 10^1$, the exponent of 10 is positive.

We do not attempt to write numbers less than zero in scientific notation.

Scientific notation is useful in comparing numbers. Which is larger

$$9.803 \times 10^3 \quad \text{or} \quad 1.2 \times 10^4 \ ?$$

At first glance it appears that the number on the left is larger since 9.803 is greater than 1.2.

But $9.803 \times 10^3 = 9{,}803$
and $1.2 \times 10^4 = 12{,}000$

Which is actually larger? Both 9.803 and 1.2 are numbers between 1 and 10 but 9.803 is not 10 times 1.2. The factor $10^4$ is 10 times $10^3$. Hence the second factor controls the size of the number.

Thus $1.2 \times 10^4 > 9.803 \times 10^3$
Similarly $4.6 \times 10^5 > 5.932 \times 10^2$
$2.3 \times 10^8 \geq 8.6 \times 10^7$
$5.962 \times 10^{-1} > 6.854 \times 10^{-2}$

When two numbers are written in scientific notation the one with the greater exponent is the greater number.

### Exercises 21-3c

1. Write each of the following in scientific notation.
   a. 0.093
   b. 0.0001
   c. $\frac{1}{10^6}$
   d. 1
   e. 0.00621
   f. $\frac{1}{10^2}$
   g. 0.7006
   h. 0.000000907
   i. 6
   j. 0.0045
2. Write each of the following in decimal notation.
   a. $9.3 \times 10^{-5}$
   b. $1.07 \times 10^{-1}$
   c. $10^{-6}$
   d. $5 \times 10^{-4}$
   e. $7.065 \times 10^{-3}$
   f. $10^{-1}$
   g. $14.3 \times 10^{-7}$
   h. $385.76 \times 10^{-6}$
3. Write each of the following in scientific notation.
   a. $63 \times 10^4$
   b. $0.157 \times 10^{-3}$
   c. 0.0000024
   d. $5.265 \times 10^{-5}$
   e. 362.35
   f. $10^{-5} \times 432$
   g. 0.00000000305
   h. $69.5 \times 10^{-1}$

4. Fill in the blanks in the following to make a true sentence. Notice that in some cases scientific notation is NOT used.

a. $0.006 = 6 \times 10^{\square}$

b. $0.000063 = \square \times 10^{-5}$

c. $0.0004015 = 4015 \times 10^{\square}$

d. $6000.0 = 0.06 \times 10^{\square}$

e. $0.213 = 2.13 \times 10^{\square}$

f. $0.213 = 213 \times 10^{\square}$

g. $0.213 = \square \times 10^{-5}$

h. $0.213 = \square \times 10^{-4}$

5. By inspection replace the "?" by one of the symbols > or <.

a. $8 \times 10^{4}$ ? $8 \times 10^{3}$

b. $8 \times 10^{-4}$ ? $8 \times 10^{-3}$

c. $8 \times 10^{7}$ ? $2 \times 10^{8}$

d. $7 \times 10^{2}$ ? $9 \times 10^{-4}$

e. $9.99 \times 10^{6}$ ? $1.01 \times 10^{7}$

f. $2.4 \times 10^{-8}$ ? $3.42 \times 10^{-9}$

g. $3 \times 10^{-2}$ ? $1.4 \times 10^{-1}$

h. $5.5 \times 10^{-3}$ ? $6 \times 10^{-2}$

i. $3 \times 10^{0}$ ? $4 \times 10^{-1}$

j. $2.79 \times 10^{2}$ ? $3.1 \times 10^{2}$

6. By inspection, tell which of the following are:

i. numbers between 0 and 1.

ii. numbers between 1 and 10.

iii. numbers greater than 10.

a. $2 \times 10^{0}$

b. $10^{4}$

c. $10^{-7}$

d. $2 \times 10^{-3}$

e. $4.5 \times 10^0$

f. $1.2 \times 10^{-1}$

g. $8.02 \times 10^3$

h. $8.02 \times 10^{-4}$

i. $1.2 \times 10^{-1}$

*j. $.12 \times 10^1$

*k. $21.43 \times 10^{-1}$

*l. $0.3 \times 10^0$

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## 21-4. Multiplication of Large and Small Numbers.

You have already multiplied numbers such as $10^3$ and $10^5$ by using addition as $10^3 \times 10^5 = 10^{3+5} = 10^8$. You also know the meaning of negative exponents. Sometimes problems like $10^{-5} \times 10^{-3}$, or $10^5 \times 10^{-3}$ arise. Suppose we study some examples.

### Exercises 21-4a

(Class Discussion)

1. Mutliply $10^{-5} \times 10^{-3}$

   a. $10^{-5} \times 10^{-3} = \frac{1}{10^5} \times \frac{1}{10^3}$ Why?

   b. $\frac{1}{10^5} \times \frac{1}{10^3} = \frac{1}{10^5 \times 10^3}$ Why?

   c. $\frac{1}{10^5 \times 10^3} = \frac{1}{10^8}$ Why?

   d. $\frac{1}{10^8} = 10^{-8}$ Why?

   e. $10^{-5} \times 10^{-3} = 10^{-8}$ From steps a to d

   f. $^-5 + (^-3) = ?$

   g. In what easy way can the final exponent, $^-8$, be found?

2. Test the conclusion you reached in 1f. on this example:

$$10^{-2} \times 10^{-4}$$

3. Consider the problem $10^5 \times 10^{^-3}$

a. $10^5 \times 10^{^-3} = 10^5 \times \frac{1}{10^3}$ Why?

b. $10^5 \times \frac{1}{10^3} = \frac{?}{?}$ Why?

c. $\frac{10^5}{10^3} = 10^?$ Why?

d. $10^5 \times 10^{^-3} = 10^?$ From steps a to c

e. In what easy way can the final exponent, 2 , be found?

4. Multiply $10^{^-5} \times 10^3$

a. $10^{^-5} \times 10^3 = \frac{1}{10^5} \times 10^3$ Why?

b. $\frac{1}{10^5} \times 10^3 = \frac{?}{?}$

c. $\frac{10^3}{10^5} = \frac{1}{?}$

d. $\frac{1}{10^2} = 10^?$

e. $10^{^-5} \times 10^3 = 10^?$ From steps a to d

f. In what easy way can the final exponent, $^-2$, be found?

5. Test your conclusions on the following:

a. $10^{^-2} \times 10^{^-1} = ?$
b. $10^4 \times 10^{^-3} = ?$
c. $10^4 \times 10^3 = ?$
d. $10^{^-2} \times 10^{^-4} = ?$
e. $10^{^-5} \times 10^2 = ?$
f. $10^{^-1} \times 10^0 = ?$

The same method of addition of exponents can be used whether the exponents are positive or negative or zero. We can use letters to say this briefly.

$$10^a \times 10^b = 10^{(a+b)}$$

where a and b are integers.

More complicated multiplication problems can be done very simply using exponents.

**Example 1:** Multiply $4.3 \times 10^5$ by $2 \times 10^{-3}$.

$$(4.3 \times 10^5) \times (2 \times 10^{-3}) = (4.3 \times 2) \times (10^5 \times 10^{-3})$$
$$= 8.6 \times 10^2$$

**Example 2:** Multiply and express your answer in scientific notation.

$$\begin{aligned}(4.7 \times 10^{-3}) \times (5.4 \times 10^7) &= (4.7 \times 5.4) \times (10^{-3} \times 10^7)\\ &= 25.38 \times 10^{(-3+7)}\\ &= 25.38 \times 10^4\\ &= (2.538 \times 10) \times 10^4\\ &= 2.538 \times (10 \times 10^4)\\ &= 2.538 \times 10^5\end{aligned}$$

### Exercises 21-4b

1. Write the following products in scientific notation.

a. $10^{-5} \times 10^{-2}$ e. $0.0001 \times 0.007$

b. $0.3 \times 10^{-2}$ f. $(5.7 \times 10^{-3}) \times 10^{-7}$

c. $10^{-7} \times 10^{-6}$ g. $10^{12} \times 10^{-3} \times 10^{15}$

d. $0.04 \times 0.002$ h. $10^{12} \times 10^{-7} \times 10^{-8}$

2. Write the following products in scientific notation.

a. $0.0012 \times 0.000024$ d. $3 \times 10^{-6} \times 10^{-4}$

b. $6 \times 10^{-7} \times 9 \times 10^{-3}$ e. $38 \times 10^{-3} \times 0.00012$

c. $14 \times 10^{-3} \times 10^{-5}$ f. $0.000896 \times 0.00635$

3. Using scientific notation find the products of the following:

a. $10{,}000 \times 0.01$ c. $10^{17} \times 10^{-23}$

b. $0.00001 \times 10{,}000{,}000$ d. $10^6 \times \frac{1}{10^4} \times \frac{1}{10^5} \times 10^{-4}$

4. Multiply forty-nine thousandths by seven and six hundredths using scientific notation. Express your answer in scientific notation.

5. A large corporation decided to invest some of its surplus money in bonds. If 11 million dollars was invested at an average annual rate of $3\frac{3}{4}\%$, what was the annual income from this investment? Use scientific notation in the computation, and also express your answer in scientific notation.

6. On a certain date the national debt, rounded to the nearest 100 billion dollars, was 300 billion dollars. Assuming that the government pays an average rate of interest of 3.313 %, what is the number of dollars in interest paid each year? Express the answer in scientific notation.

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## 21-5. Division of Large and Small Numbers.

The principles involved in division are much like those you have been using. You are familiar with the division of $10^6$ by $10^4$. The definition of $10^{-n}$ as $\frac{1}{10^n}$, where n is a counting number, leads immediately to the solution of the following example:

$$10^6 \div 10^{-3} = \frac{10^6}{10^{-3}} = \frac{10^6}{\frac{1}{10^3}} = 10^6 \times \frac{10^3}{1} = 10^9$$

### Exercises 21-5a
(Class Discussion)

1. Using ideas similar to those used above, perform the following:
   a. $10^{-4} \div 10^5$
   b. $10^{-2} \div 10^{-7}$
2. a. Divide $10^7$ by $10^2$.
   b. Why is $10^{7-2}$ equal to $10^5$ ?
   c. Are $10^7 \div 10^2$ and $10^{7-2}$ numerals for the same number, or different numbers?

3. a. Find $6 - (^-3)$

   b. Find $10^{6-(^-3)}$

   c. Use the illustrative example above to determine whether $10^6 \div 10^{^-3}$ and $10^{6-(^-3)}$ are numerals for the same number.

4. a. Find $10^{^-4-5}$.

   b. Is $10^{^-4} \div 10^5$ equal to $10^{(^-4-5)}$? Why? (You found the first number in 1(a).)

5. Is $10^{^-2} \div 10^{^-7} = 10^{(^-2-^-7)}$?

The pattern that you developed in the exercises above is always true. In general,

$$10^a \div 10^b = \frac{10^a}{10^b} = 10^{(a-b)}$$

where a and b are integers.

This result allows you to do a subtraction problem instead of a division problem when you are dividing powers of 10. For example:

$$10^{11} \div 10^{^-5} = 10^{11-(^-5)} = 10^{16}$$

$$10^{^-8} \div 10^{^-9} = 10^{^-8-(^-9)} = 10$$

$$10^{^-3} \div 10^{9} = 10^{^-3-9} = 10^{^-12}$$

Now extend the method to a more complicated problem:

$$(3.6 \times 10^{^-4}) \div (2 \times 10^8) = \frac{3.6 \times 10^{^-4}}{2 \times 10^8} \qquad \text{Why?}$$

$$= \frac{3.6}{2} \times \frac{10^{^-4}}{10^8} \qquad \text{Why?}$$

$$= 1.8 \times 10^{^-12} \qquad \text{Why?}$$

### Exercises 21-5b

1. Write in scientific notation.

   a. $10^5 \div 10^2$

   b. $10^3 \div 10$

   c. $10^{14} \div 10^4$

   d. $10^{17} \div 10^{12}$

e. $10^{11} \div 10^{13}$ g. $10^{6} \div 10^{12}$

f. $10^{10} \div 10^{20}$ h. $10^{3} \div 10^{4}$

2. Write in scientific notation.

a. $10^{5} \div 10^{-2}$ e. $10^{11} \div 10^{-13}$

b. $10^{3} \div 10^{-1}$ f. $10^{10} \div 10^{-20}$

c. $10^{14} \div 10^{-4}$ g. $10^{6} \div 10^{-12}$

d. $10^{17} \div 10^{-12}$ h. $10^{3} \div 10^{-4}$

3. Write in scientific notation.

a. $10^{-5} \div 10^{2}$ e. $10^{-11} \div 10^{13}$

b. $10^{-3} \div 10$ f. $10^{-10} \div 10^{20}$

c. $10^{-14} \div 10^{4}$ g. $10^{-6} \div 10^{12}$

d. $10^{-17} \div 10^{12}$ h. $10^{-3} \div 10^{4}$

4. Write in scientific notation.

a. $10^{-5} \div 10^{-2}$ e. $10^{3} \div 10^{-1}$

b. $10^{-14} \div 10^{-4}$ f. $10^{-17} \div 10^{-12}$

c. $10^{-11} \div 10^{-13}$ g. $10^{-10} \div 10^{-20}$

d. $10^{-6} \div 10^{-12}$ h. $10^{-3} \div 10^{-4}$

5. Write in scientific notation.

a. $(6 \times 10^{-5}) \div (3 \times 10^{-2})$

b. $(7 \times 10^{-3}) \div 10^{4}$

c. $(1.2 \times 10^{6}) \div 10^{-3}$

d. $(2.4 \times 10) \div 10^{-1}$

e. $\dfrac{9.6 \times 10^{-4}}{2.4 \times 10^{2}}$

f. $\dfrac{7.6}{1.9 \times 10^{3}}$

6. Fill in the blank spaces with the proper symbol.

a. $.12 = \dfrac{12}{100} = \dfrac{12}{10^{2}} = 12 \times 10^{\boxed{\phantom{00}}} = 1.2 \times 10^{\boxed{\phantom{00}}}$

b. $.46 = \dfrac{46}{100} \quad \dfrac{46}{10^{2}} = 46 \times 10^{\boxed{\phantom{00}}} = 4.6 \times 10^{\boxed{\phantom{00}}}$

c. $0.03 = \frac{3}{100} = \frac{3}{10^{\boxed{\phantom{0}}}} = 3 \times 10^{\boxed{\phantom{0}}} = 3 \times 10^{\boxed{\phantom{0}}}$

d. $.350 = \frac{350}{1000} = 350 \times 10^{\boxed{\phantom{0}}} = 3.5 \times 10^{\boxed{\phantom{0}}}$

e. $\frac{450}{30} = \frac{450}{3 \times 10} = 150 \times 10^{\boxed{\phantom{0}}} = 1.5 \times 10^{\boxed{\phantom{0}}}$

f. $\frac{4800}{24\%} = \frac{4800}{2.4 \times 10^{\boxed{\phantom{0}}}} = \frac{4.8 \times 10^{\boxed{\phantom{0}}}}{2.4 \times 10^{\boxed{\phantom{0}}}} = 2 \times 10^{\boxed{\phantom{0}}}$

7. A city government has an income of \$2,760,000 for this year. The income this year represents 3 % of the total value of taxable property. What is the total value of taxable property? Use scientific notation in your computations.

8. A commuter pays \$.40 per day for his fare. Is it reasonable to expect that he will spend one million cents in fares before he retires? Assume that he travels 250 days per year.

*9. At the rate of ten dollars per second, about how many days will it take to spend a billion dollars? Assume that this goes on 24 hours a day. ( 1 day = $8.64 \times 10^4$ seconds.)

*10. The tax raised in a certain county is \$160,000 on an assessed valuation of \$8,000,000. If Mr. Smith's tax is \$400, what is the assessed value of his property?

*11. It costs about \$35,000,000 to equip an armored division and about \$14,000,000 to equip an infantry division. The cost of equipping an infantry division is what percent of the cost of equipping an armored division?

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## 21-6. Use of Exponents in Multiplying and Dividing Decimals.

You are familiar with the operations of multiplying and dividing two numbers in decimal form. The reason for the placement of the decimal point can be explained by using exponents.

Below are three multiplication problems:

a) $3214 \times 16$

b) $32.14 \times 1.6$

c) $0.03214 \times 0.16$

Each problem is different but (b) and (c) are very closely related to (a). We can see the connection by using exponents.

$$\begin{aligned} \text{b)}\quad 32.14 \times 1.6 &= (3214 \times 10^{-2}) \times (16 \times 10^{-1}) \\ &= (3214 \times 16) \times (10^{-2} \times 10^{-1}) \\ &= (3214 \times 16) \times 10^{-3}. \end{aligned}$$

$$\begin{aligned} \text{c)}\quad 0.03214 \times 0.16 &= (3214 \times 10^{-5}) \times (16 \times 10^{-2}) \\ &= (3214 \times 16) \times (10^{-5} \times 10^{-2}) \\ &= (3214 \times 16) \times 10^{-7} \end{aligned}$$

Each problem involves multiplying the whole number 3214 by the whole number 16. For this part of the computation you use the form:

$$\begin{array}{r} 3214 \\ \underline{16} \\ 1\,9284 \\ \underline{3214\phantom{0}} \\ 51424 \end{array}$$

Example (a) is finished when the product 51424 is found.

To find the product in (b) one more step is necessary: we have to multiply $51424 \times 10^{-3}$.

Since $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = \frac{1}{10 \times 10 \times 10} = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}$, to multiply by $10^{-3}$ means to multiply three times by $\frac{1}{10}$ (or divide three times by 10). Each time that we multiply 51424 by $\frac{1}{10}$ we make the place value of each digit $\frac{1}{10}$ as large. This has the effect of moving the digits a place to the right. Three such multiplications by $\frac{1}{10}$ shift the digits three places to the right.

$$51424. \times 10^{-3} = 51.424$$

To find the product in (c) we have a similar second step, to multiply $51424 \times 10^{{}^-7}$ and so we move the digits seven places to the right making their place values smaller.

$$51424. \times 10^{{}^-7} = 0.0051424$$

## Exercises 21-6a
(Class Discussion)

Use the above procedure to find each of the following products.

1. $6.14 \times 0.42$
2. $0.625 \times 0.038$
3. $649.3 \times 14.68$
4. $11.4 \times 0.0031$

Now consider division. Let us divide 14.72 by 6.1.

$$\frac{14.72}{6.1} = \frac{1472 \times 10^{{}^-2}}{61 \times 10^{{}^-1}} = \frac{1472}{61} \times \frac{10^{{}^-2}}{10^{{}^-1}} = \frac{1472}{61} \times 10^{({}^-2 - {}^-1)}$$

$$= \frac{1472}{61} \times 10^{{}^-1}.$$

First we divide 1472 by 61. This is an operation on whole numbers. The result of this operation is 24.13. This part of the quotient is correct to two decimal places. But we must multiply 24.13 by $10^{{}^-1}$.

Hence,

$$\frac{14.72}{6.1} = 24.13 \times 10^{{}^-1} = 2.413.$$

The final answer for this problem is now correct to three decimal places. Once again the exponent ${}^-1$ is used to fix the position of the decimal point in the answer. Note that powers of ten were used in such a fashion that actually we were dividing one whole number by another whole number.

## Exercises 21-6b

1. Place the decimal point in the products to make the following number sentences true.

   a. $6021 \times 0.00003 = (6021) \times (3 \times 10^{{}^-5}) = 18063$

b. $3.42 \times 0.02 = (342 \times 10^{-2}) \times (2 \times 10^{-2}) = 684$

c. $2.5 \times 3{,}000 = (25 \times 10^{-1}) \times (3 \times 10^{3}) = 75$

d. $54.73 \times 7.3 + (5473 \times 10^{-2}) \times (73 \times 10^{-1}) = 399529$

e. $1200 \times 0.006 = (12 \times 10^{2}) \times (6 \times 10^{-3}) = 72$

2. Fill the blanks with proper symbols.

a. $4.52 = 45.2 \times 10^{-1} = 452 \times 10^{\square}$

b. $0.012 = 1.2 \times 10^{-2} = 12 \times 10^{\square}$

c. $65000 = 6.5 \times 10^{\square} = 65 \times 10^{3}$

d. $38.216 = 382.16 \times 10^{-1} = 3821.6 \times 10^{-2} = 38216 \times 10^{\square}$

e. $6.37 \times 10^{4} = 63.7 \times 10^{3} = 637 \times 10^{2} = \square \times 10^{0}$

f. $0.003 \times 10^{5} = 3 \times 10^{\square} = 30 \times 10^{\square}$

g. $41.2 \times 10^{-3} = 0.412 \times 10^{-1} = \boxed{0.\quad} \times 10^{0}$

3. Complete the quotients to make the following sentences true.

a. $\dfrac{6004}{0.02} = \dfrac{6004 \times 10^{0}}{2 \times 10^{-2}} = 3002 \times 10^{2} = 3002$

b. $\dfrac{0.366}{0.06} = \dfrac{366 \times 10^{-3}}{6 \times 10^{-2}} = 61 \times 10^{-1} = 61$

c. $\dfrac{0.084}{12000} = \dfrac{84 \times 10^{-3}}{12 \times 10^{3}} = 7 \times 10^{-6} = 7$

d. $\dfrac{0.2}{0.00125} = \dfrac{2 \times 10^{-1}}{125 \times 10^{-5}} = 0.016 \times 10^{4} = 16$

## Exercises 21-6c

1. Multiply using exponent notation. Give answers in decimal notation.

a. $135 \times 0.06 =$

b. $76{,}000 \times 3{,}000 =$
Hint: $(76 \times 10^{3}) \times (3 \times 10^{3})$

c. $18{,}000 \times 0.0003 =$

d. $0.0035 \times 16.301 =$

e. $6,000,000 \times 0.0275 =$ f. $0.07 \times 300 \times 0.02 \times 6,000 =$

2. Divide using exponent notation. Give answers in decimal notation.

a. $6.3 \div 0.3$

b. $0.78 \div 13$

c. $\frac{8750}{8.75}$

d. $\frac{0.1470}{0.75}$

e. $0.27\overline{)0.84402}$

f. $1800\overline{)21.6}$

*3. How many pieces of popcorn each weighing 0.04 ounce does it take to fill 840 bags? Each bag contains 6 ounces of popcorn.

4. BRAINBUSTER. A space rocket can travel at 100,000 miles per second. About how long (in years) will it take it to visit and return from a star that is $5\frac{1}{3}$ light years away?

1 light year $\approx 6.3 \times 10^{12}$ miles

1 year $\approx 3.2 \times 10^{7}$ seconds

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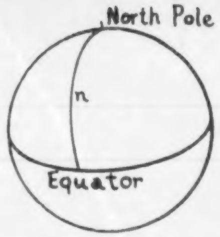
## 21-7. The Metric System.

The system of measures which is used most widely in the United States is called the English system. Some of the units in this system are the inch, foot, yard, mile, pound, etc. In most other countries, the system of measures used is the metric system in which the basic unit of linear measure is the meter. In the year 1789 a group of French mathematicians were called together to develop a simplified system of weights and measures. They decided that since their system of numeration was a decimal (base ten) system, they should have a decimal basis for their system of measures. In such a system there are basic units of length, area, volume, etc. Other units are a power of ten times the basic unit. With such a system it is easy to convert from one unit to another by multiplying or dividing by a power of ten.



The French mathematicians calculated the distance from the North Pole to the equator on the meridian through Paris. They took $\frac{1}{10,000,000}$ of this distance for their unit of length. By defining the unit this way, they made it possible for the original distance to be determined again if the standard bar of unit length were ever lost. They named the unit of length the meter. Later this definition was improved.

$$1 \text{ meter} = \frac{n}{10,000,000}$$

A meter is a little longer than a yard. It is approximately equal to 39.37 inches.

Other units of length in the metric system are as follows: For shorter distances,

$$1 \text{ decimeter} = \frac{1}{10} \text{ of a meter} = 0.1 \text{ meter.}$$

$$1 \text{ centimeter} = \frac{1}{10} \text{ of a decimeter or } \frac{1}{10} \times \frac{1}{10} \text{ of a meter}$$

$$= \frac{1}{100} \text{ of a meter} = 0.01 \text{ meter}$$

$$1 \text{ millimeter} = \frac{1}{10} \text{ of a centimeter or}$$

$$\frac{1}{10} \times (\frac{1}{100} \text{ of a meter}) = \frac{1}{1000} \text{ of a meter}$$

$$= 0.001 \text{ meter.}$$

For longer distances,

1 dekameter = 10 meters

1 hectometer = 10 dekameters or $10 \times 10 = 10^2$ meters

1 kilometer = 10 hectometers or $10 \times 10 \times 10 = 10^3$ meters.

Many attempts have been made to persuade the United States Government to adopt the metric system for general use. Thomas Jefferson in the Continental Congress worked for a decimal system of money and measures but succeeded only in securing a decimal system of money. When John Quincy Adams was Secretary of State, he foresaw world metric standards in his 1821 "Report on Weights and Measures." In 1866, Congress authorized the use of the metric system, making it legal for those who wished to use it. Finally, in 1893, by act of Congress, the meter was made the standard of length in the United States. The yard and the pound are now officially defined in terms of the metric units, the meter and the kilogram.

### Exercises 21-7a
(Class Discussion)

1. What advantages does the metric system have over the systems we use?
2. What difficulties would probably arise if we replaced our system by the metric system?

The following table shows the plan of linear metric units. Notice how useful the exponent notation is, in showing relationships in the metric system. Abbreviations for the unit names appear in the parentheses following each unit of length.

1 millimeter (mm.) = $10^{-3}$ meter
1 centimeter (cm.) = $10^{-2}$ meter
1 decimeter (dm.) = $10^{-1}$ meter
1 meter (m.) = $10^{0}$ meter
1 dekameter (dkm.) = $10^{1}$ meters
1 hectometer (hm.) = $10^{2}$ meters
1 kilometer (km.) = $10^{3}$ meters

Notice that all the names of the metric units (except "meter") use the word "meter" with a prefix. It is important to remember these prefixes, because they are used also to name other units of measure in the metric system.

| Prefix | Meaning |
|---|---|
| kilo | 1000 or $10^{3}$ |
| hecto | 100 or $10^{2}$ |
| deka | 10 or $10^{1}$ |
| deci | $\frac{1}{10}$ or $10^{-1}$ |
| centi | $\frac{1}{100}$ or $10^{-2}$ |
| milli | $\frac{1}{1000}$ or $10^{-3}$ |

## Exercises 21-7b

1. **Tell how you change from a unit to the next larger unit in the metric system.**

2. **Tell how you change from a unit to the next smaller unit in the metric system.**

3. Complete each of the following:
   a. 1 kilometer = ____________ hectometers
   b. 1 kilometer = ____________ dekameters
   c. 1 kilometer = ____________ meters
   d. 1 kilometer = ____________ decimeters
   e. 1 kilometer = ____________ centimeters
   f. 1 kilometer = ____________ millimeters

4. Fill in each blank with the correct numeral.
   a. 5 m. = ________cm.
   b. 200 cm. = _____mm.
   c. 500 m. = ______km.
   d. 2.54 cm. = ____mm.
   e. 1.5 km. = _____m.
   f. 3.25 m. = ________cm.
   g. 3500 m. = ________km.
   h. 474 cm. = ________m.
   i. 5.5 cm. = ________mm.
   j. 6.25 m. = ________cm.

*5. A meter was originally defined to be $\frac{1}{10,000,000}$ of the distance on the earth's surface from the North Pole, to the equator. Using scientific notation express the numb[illegible]f:
   a. meters in the circumference of the earth.
   b. dekameters in the circumference of the earth.
   c. hectometers in the circumference of the earth.

With the recent advances in atomic theory even smaller and larger units are necessary in the metric system. The following are now in use:

| Prefix | Meaning |
|---|---|
| mega | $1,000,000 = 10^6$ |
| micro | $\frac{1}{1,000,000} = 10^{-6}$ |

You may have read in the newspapers or heard on the radio the term megacycle (1 million cycles). Another term which you probably are familiar with is micron ($10^{-6}$ m.). Have you heard of other terms using the prefix mega or micro?

### Exercises 21-7c

Complete each of the following:

1. 11.11 meters = ________ km.
2. 5.34 meters = ________ cm.
3. 24.5 meters = ________ km.
4. .52 meters = ________ mm.
5. 643.2 meters = ________ km.
6. 202.2 meters = ________ cm.
7. .015 mm. = ________ microns.

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## 21-8. Conversion to English Units.

Since we use the English and metric systems in this country it is often necessary to convert from one system to another. This is very important in many areas, such as foreign trade and foreign investments. To convert from one system to another, we use the following facts:

$$1 \text{ inch} = 2.54 \text{ cm. (exactly)}$$

and

$$39.37 \text{ in.} \approx 1 \text{ meter}$$

In terms of yards, this says that

$$1 \text{ meter} \approx \frac{39.37}{36} \text{ yards, or}$$

$$1 \text{ meter} \approx 1.1 \text{ yards.}$$

Since one inch is equal to 2.54 centimeters, the ratio of inches to centimeters is

$$\frac{1}{2.54}\,.$$

Suppose that we want to change 5 inches to centimeters. We want to find $x$, the number of centimeters in 5 inches. Then the ratio of 5 to $x$ will be the same as 1 to 2.54. This may be expressed by the proportion

$$\frac{\text{number of inches}}{\text{numbers of centimeters}} = \frac{1}{2.54} = \frac{5}{x}$$

$$1 \cdot x = 5 \cdot (2.54)$$

$$x = 12.70$$

5 inches is equivalent to 12.70 cm.

## Exercises 21-8a

1. Express 9 inches as centimeters.

   Using the ratio of inches to centimeters, we have

$$\frac{1}{2.54} = \frac{9}{x}$$

2. Change 15 inches to centimeters.

$$\frac{1}{2.54} = \frac{?}{?}$$

How do we change from centimeters to inches? Again our constant ratio is $\frac{1}{2.54}$.

Let us change 15.24 centimeters to inches. Call $y$ the number of inches in 15.24 centimeters. Then form the ratio of $y$ to 15.24 and the proportion:

$$\frac{1}{2.54} = \frac{y}{15.24}$$

$$2.54y = 15.24$$

$$y = \frac{15.24}{2.54}$$

$$y = 6 \qquad \text{(6 inches)}$$

3. How can you change 27.94 centimeters to inches?

$$\frac{?}{?} = \frac{y}{27.94}$$

4. Change 12.70 centimeters to inches.

$$\frac{1}{2.54} = \frac{?}{?}$$

5. Can you change 5 feet to centimeters? Using the ratio of inches to centimeters, what is the first step?

$$\frac{1}{2.54} = \frac{?}{x}$$

6. How would you change 9 yards to centimeters?

$$\frac{1}{2.54} = \frac{?}{?}$$

## Exercise 21-8b

1. Change each of the following English measurements to centimeters.

   a. 8 inches
   b. 24 inches
   c. 19 inches
   d. 27 inches
   e. 9 feet
   f. 1 yard
   g. 5 yards
   h. 24 feet

2. Change each of the following metric measurements to inches.

   a. 5.08 cm.
   b. 35.56 cm.
   c. 17.78 cm.
   d. 21.59 cm.
   e. 30.48 cm.
   f. 38.10 cm.
   g. 7.62 cm.
   h. 13.97 cm.

3. Change each of the following metric measurements to the nearest yard.

   a. 100 meters
   b. 200 meters
   c. 400 meters
   d. 800 meters
   e. 1500 meters

4. Convert each of the following metric measurements to miles.

   a. 10 kilometers
   b. 50 kilometers
   c. 100 kilometers
   d. 500 kilometers
   e. 1000 kilometers

5. Mount Everest is approximately 29,000 feet high. Express this number in meters.

6. The distance from the earth to the sun is about 93 million miles. What is this distance in kilometers?

7. A common size of typing paper is $8\frac{1}{2}$ inches by 11 inches. What are these dimensions in centimeters?

8. What is your height in
   a. millimeters?
   b. centimeters?
   c. microns?

---

## 21-9. Summary.

A number is expressed in scientific notation if it is written as the product of a number between 1 and 10 and the proper power of ten, or as a power of 10.

$10^0$ is defined as 1 .

If $n$ is a counting number, then $10^{-n}$ means $\frac{1}{10^n}$ .

When $a$ and $b$ are integers, we have:

$$10^a \times 10^b = 10^{(a+b)}$$

When $a$ and $b$ are integers, we have:

$$\frac{10^a}{10^b} = 10^{(a-b)}$$

In the United States, we usually use the English system of measurement, but in most parts of the world, the metric system is used.

Conversions between the metric system and the English system can be performed by solving proportions using the following facts:

1 inch = 2.54 cm.

39.37 inches ≈ 1 meter

---

## 21-10. Chapter Review.

### Exercises 21-10

1. Using an exponent, write the following.
   a. 500
   b. 5000
   c. 50,000
   d. 567,000
   e. 567,000,000
   f. 567,000,000,000

2. Write a numeral for each of the following in a form which does not use an exponent or indicate a product.
   - a. $10^4$
   - b. $4.83 \times 10^2$
   - c. $3 \times 10^5$
   - d. $9.3 \times 10^2$
   - e. $536 \times 10^6$
   - f. $15.3 \times 10^5$

3. Multiply and express your answer in scientific notation.
   - a. $10^2 \times 10^3$
   - b. $6 \times 10^6 \times 10^3$
   - c. $10^{12} \times 10 \times 10^4$
   - d. $10^0 \times 10^0$
   - e. $10 \times 10^0$
   - f. $10^2 \times 10^8$

4. Express each of the following using negative exponents.
   - a. $\frac{1}{1000}$
   - b. $\frac{1}{10^2}$
   - c. $\frac{1}{10^9}$
   - d. $\frac{1}{10,000}$
   - e. $\frac{1}{1,000,000,000}$
   - f. $\frac{1}{10^{19}}$

5. Express each of the following as a fraction.
   - a. $10^{-1}$
   - b. $10^{-7}$
   - c. $10^{-28}$
   - d. $10^{-10}$
   - e. $10^{-297}$
   - f. $10^{-792}$

6. Express the following products in scientific notation.
   - a. $0.1 \times 10^{-2}$
   - b. $0.04 \times 0.003$
   - c. $10^{12} \times 10^{-6} \times 10^{-9}$
   - d. $2 \times 10^{-6} \times 10^{-3}$
   - e. $10,000 \times 10$
   - f. $100,000 \times 1,000,000,000$

7. Write the answers using scientific notation.
   - a. $10^4 \div 10^{-2}$
   - b. $10^{14} \div 10^{-2}$
   - c. $10^3 \div 10^{-4}$
   - d. $10^{10} \div 10^5$
   - e. $10^9 \div 10^0$
   - f. $10^{27} \div 10^{-27}$

8. Fill in each blank with the correct numeral.
   a. 4m. = ________cm.
   b. 100 cm. = ________mm.
   c. 500 m. = ________km.
   d. 2.54 cm. = ________m.
   e. 5.4 cm. = ________mm.
   f. 5.25 m. = ________cm.

---

## 21-11. Cumulative Review.

### Exercises 21-11

1. At the end of a game consisting of 5 rounds of play, Oscar's score is 23 and Tom's score is $^-15$. What is the difference in their scores?

2. Express as a fraction and then simplify:
   a. $\frac{2}{3} + \frac{3}{4}$
   b. $\frac{2}{3} \cdot \frac{3}{4}$
   c. $\frac{2}{3} \div \frac{3}{4}$
   d. $\frac{2}{3} \cdot \frac{2}{3}$
   e. $\frac{0}{3} \cdot \frac{3}{2}$
   f. $\frac{3}{2} \div \frac{3}{2}$
   g. $\frac{3}{2} - \frac{4}{3}$
   h. $\frac{4}{3} - \frac{3}{2}$

3. Insert one of the symbols >, <, or = between $\frac{13}{42}$ and $\frac{17}{56}$ so as to make a true statement.

4. Find the sum of 3, $^-7$, 12, $^-1$, 0, 17, $^-12$, and $^-16$.

5. Find the value of
   a. $(^-1.5) \times 8$
   b. $9 \times {}^-0.4$
   c. $^-6 \times \frac{^-1}{2}$
   d. $^-9 - (^-4)$
   e. $2 \cdot (^-3 + {}^-5)$
   f. $^-1 \cdot (4 + {}^-7)$
   g. $\frac{^-1}{2} \div 4$
   h. $\frac{^-5}{8} \div \frac{^-2}{8}$

6. Find the value of x which makes each of the following sentences true.

a. $2x = 18$

b. $x - 13 = {}^{-}2$

c. $7x = 3$

d. $11x = 0$

e. $13 + x = {}^{-}1$

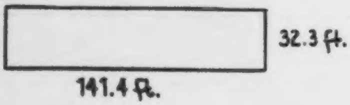
f. $11x = 24$

7. How much time is there between 7:35 a.m. and 3:10 p.m.?

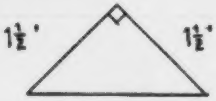
8. How many square yards of area does a chalkboard 4 feet by $8\frac{1}{2}$ feet contain?

9. Find the area of the following figures:

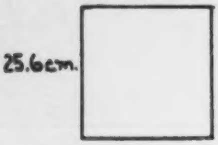
a. a rectangle



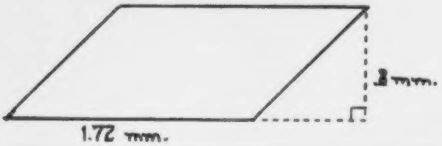
c. a right triangle



b. a square



d. a parallelogram



10. The circumference of a bicycle wheel is 75 in. What is the length of one of its spokes? (Use $\pi = 3.14$.)

11. If $3\frac{1}{3}$ million comic books are sold every day in this country, are more or less than a billion comic books sold every year?

# INDEX

acute angle, 92
addition
  of negative numbers, 247-249, 252-254
  of rational numbers on the number line, 247-249, 252-254
additive inverse, 254, 273, 373, 374
adjacent angles, 81
altitude of a cylinder, 193
angle, 75
  acute, 92
  adjacent, 81
  congruent, 76
  corresponding, 96
  measure of, 75, 77
  non-adjacent, 83
  obtuse, 92
  right, 92
  supplementary, 85
  vertical, 83, 87
"approximately equal to", 380
approximation, 47
arc of circle, 179, 201
area, 33
  of a circle, 189, 201
  of a closed region, 33
  cutting units of, 36
  meaning of, 33
  of a parallelogram, 139
  of a prism, 51
  of a rectangle, 36, 43, 68
  of a triangle, 145
  unit of, 34
arithmetic mean, 223
associative property
  of addition, 273, 373, 374
  of multiplication, 267, 273, 373, 374
averages, 223
  arithmetic mean, 223
  median, 225
  mode, 227
axes, 307
axis, 307
bar graphs, 208
broken-line graphs, 215
centimeter, 404, 405
central angle, 180, 201

circle
    arc of, 179
    area of, 187
    central angle of, 180
    circumference of, 182, 184
    concentric, 174
    definition of, 166, 169
    diameter, 175
    exterior of, 170
    interior of, 170
    point of tangency, 177
    radius, 166
    tangent to, 177
circle graphs, 219
circumference, 182, 184, 201
closure, 273, 373, 374
commutative property
    of addition, 273, 373, 374
    of multiplication, 262, 267, 273, 373, 374
compass, use of, 165, 167
completeness, 374, 375
concentric circles, 174
cone, 62
congruent, 2
congruent angles, 76
converse, 111
coordinate geometry, 316
coordinate pair, 350
coordinates in the plane, 301, 306
corresponding angles, 96
counting numbers, 245, 374
cube, 53
cubic inch, 54, 69
cubit, 9
cylinder, 62, 192, 201
    altitude, 193
    lateral area, 198, 288
    right, 192
    surface area, 198, 199
    volume, 195
cylindrical solids, 192
data, 205
    grouping, 229
decimal approximations for square roots, 354-355
decimal system, 403
decimals
    use of exponents in multiplying and dividing, 399-401
decimeter, 404, 405
degree, 76
dekameter, 404, 405
Descartes, René, 316
diameter
    definition 175, 200
    relation to circumference, 185
dimension, 64
    one-dimensional, 64
    two-dimensional, 65
    three-dimensional, 65

distance, 131
  between two parallel lines, 132
  between two points, 330-334, 343-347
  to a line, 131
distributive property, 267, 273, 373, 374
division
  by powers of ten, 386
  of negative numbers, 268-270
  of rational numbers, 268-270
English system of measurement, 403
  conversion to metric units, 407-408
equation, 279, 284, 296, 318
equilateral triangle, 108
error, greatest possible, 19
exponent, 360, 387, 388, 390-391, 394, 397, 399-401
  negative exponents, 388
exterior of circle, 170
fathom, 9
foot, 8
formulas, 287-288, 297
frequency distribution, 229
graphs, 208, 292, 297, 318
  bar, 208
  broken-line, 215
  circle, 219
  graphing solution sets of sentences, 292-294
  graphs in the plane, 316-320
greater than, 243
half-line, 245
  negative, 245
  positive, 245
half-plane, 311
  left half-plane, 311
  lower half-plane, 311
  right half-plane, 311
  upper half-plane, 311
hand, 8
hectometer, 404, 405
hexagon, 123
  regular, 176
hypotenuse, 336
identity element
  for addition, 273, 373, 374
  for multiplication, 267, 273, 373, 374
inequality, 284, 296
infinite decimals, 366-370, 374, 376
integers, 245, 374
  negative, 245
  non-negative, 245
  positive, 245
  set of, 245
interior of circle, 170
intersection of circles, 171
inverse
  additive, 254, 273, 373, 374
  multiplicative, 271, 273, 373, 374
irrational numbers, 362-365, 366-370, 376
  and infinite decimals, 366-370

isosceles triangle, 107
  angles in, 114
kilometer, 404, 405
lateral surface of a cylinder, 198, 288
left half-plane, 311
length, 5
  of a circle, 181
  same length, 2
  standard units of, 8
  unit of, 2
less than, 243
light year, 403
line
  distance between two points on a line, 330-332
  through the origin, 318, 323-327
lines in a plane, 80, 94
  intersecting, 80, 94
  parallel, 80, 99, 101
  skew, 80
lower half-plane, 311
mean, 223
measure, 5
measure of an angle, 75, 77
median, 225
mega, 406
megacycle, 406
meter, 9, 403, 404, 405
metric system, 403-408
  conversion to English units, 407-408
micro, 406
micron, 406
millimeter, 404, 405
mode, 227
multiplication
  by powers of ten, 383-384
  of negative numbers, 261-267
  of rational numbers, 260-267
  of rational numbers on the number line, 260-262
multiplication property of 0, 267, 373, 374
multiplicative inverse, 271, 273, 373, 374
negative
  exponents, 388
  half-line, 245
  integers, 245
  numbers, 244
  rational numbers, 239, 245
non-negative integers, 245
non-negative rationals, 374
number, $\pi$, 185
number line, 242-262, 357-359, 360-361, 374, 376
  addition on the, 247-249, 252-254
  multiplication on the, 260-262
  subtraction on the, 255-258

numbers
  irrational, 362-365, 366-370, 376
  negative, 244
  positive, 243, 245
  rational, 245, 272, 358, 366, 373, 374
  real, 357-359, 365, 374, 376
obtuse angle, 92
opposite, 245, 254
order property, 373
origin, 242, 306, 307
pace, 9
parallel lines, 80, 99, 101
parallelogram, 125
  altitude of, 139
  angles of, 131
  area of, 139
  bases of, 140
  opposite sides of, 128
pentagon, 123
perimeter, 21
perpendicular, 90
phrases, 279, 280, 296
pi (π), 185
  approximations of, 185
point of tangency, 177
polygons, 21, 123
  hexagon, 123
  pentagon, 123
  quadrilateral, 123
positive, 243
  half-line, 245
  integers, 245
  number, 243, 245
  rational number, 245
powers of ten
  dividing by, 386
  multiplying by, 383-384
precision, 17
prism, 148
  bases, of, 149
  edge of, 69
  faces of, 49, 68, 149
  height of, 151
  pentagonal, 149
  rectangular, 49, 68, 148
  surface area, 69
  triangular, 62, 149
  vertices of, 50, 69, 149
  volume of, 152
product of two negative numbers, 262-266
proof, 135, 138
properties of number systems, 373-375
protractor, 76
Pythagoras, 337
Pythagorean Property, 337, 344, 347
  proof of, 338-342
quadrants, 312

quadrilateral, 123
radius, 166, 175, 200
    definition, 166
    relation to circumference, 185
    relation to diameter, 175
range, 227
rational numbers, 245, 272, 358, 366, 373, 374
    addition of, 247-249, 252-254
    division of, 268-270
    multiplication of, 260-267
    negative, 239, 245
    non-negative, 374
    positive, 245
    properties of, 273, 373, 374
    set of, 245
    subtraction of, 254-258
real numbers, 357-359, 365, 374, 376
reciprocal, 271
rectangle, 21
rectangular region, 38
rectangular solid, 69
repeating decimals, 366, 370, 376
right angle, 92
right half-plane, 311
right prisms, 148
right triangle, 337, 338
ruler, 3
scalene triangle, 108
scientific notation, 379, 382, 390, 391
    definition, 382
    division, 396-397
    multiplication, 393-395
semi-circle, 180, 201
sentences, 279, 283-286, 292, 296
separation, on a circle, 179
set
    of integers, 245
    of rational numbers, 245
solution set, 286, 292-294, 296
solutions
    of equations, 279, 285
    of sentences, 279, 285
sphere, 62
square
    centimeter, 39
    foot, 39
    inch, 39, 68
    region, 39
    yard, 39
square root, ($\sqrt{\phantom{x}}$), 345
    table of, 354-355
subtraction
    of negative numbers, 255-258
    of rational numbers on the number line, 254-258
supplementary angles, 85
surface area of a cylindrical solid, 198

symbols
    > (greater than), 243, 284
    < (less than), 243, 284
    = (equal to), 284
    ≈ (approximately equal to), 380
    π (pi), 185
    √‾ (square root), 345
table--squares and square roots of numbers, 354-355
tables for reference, 73
tangent lines, 177, 201
transversal, 96, 101
triangle, 107
    altitude of, 145
    area of, 145
    base of, 146
    equilateral, 108
    isosceles, 107
    right, 337, 338
    scalene, 108
    sum of the measures of the angles of, 118
    vertices of, 107
union of circles, 171
upper half-plane, 311
"verbs", 284
    =, 284
    >, 284
    <, 284
vertical angles, 83, 87
vertices of a triangle, 107
volume
    meaning of, 53
    of cylindrical solid, 195
    of prism, 152
    of rectangular solid, 56
whole numbers, 245, 374
X-axis, 307
X-coordinate, 308, 310
    negative, 311
    positive, 311
Y-axis, 307
Y-coordinate, 308, 310
    negative, 311
    positive, 311
yard, 8
zero, 245
zero property of multiplication, 267, 373, 374

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